Fall 2021, Math 596: Week 11 Problem Set
Catch-up Week

Discussion problems. The problems below should be worked on in class.

(D1) *Practice with Ehrhart series.* Consider the polytope

$$P = \operatorname{conv}\{(1,0,0),(0,1,0),(0,-1,-1),(0,0,1),(0,0,-1)\}.$$

(a) Triangulate P using 2 simplices. Use cones over simplices to obtain $\operatorname{Ehr}_P(z)$.
Hint: what is the issue with points with negative entries? How can we handle this?

(b) Find the leading coefficient of $L_P(t)$.
Hint: this can be done without finding $L_P(t)$ in its entirety!

(D2) *Practice with V- and H-descriptions.* Refer to the polytope P from Problem (D1).

(a) Determine, using the definition of convex hull, which points in the convex hull above can be omitted. Conclude you have obtained the irredundant V-description of P.

(b) Find the irredundant H-descriptions of P. Identify a facet for each inequality in your H-description (thus ensuring irredundancy).

(c) Verify from first principles that your V- and H-descriptions are indeed equivalent.

(D3) *Practice with power series manipulation.* Consider the series

$$\sum_{n\geq 0}(n^3+2n^2+2n+1)z^n = \frac{Q(z)}{(1-z)^{d+1}} \qquad \text{and} \qquad \sum_{n\geq 0} f(n)z^n = \frac{1+3z+4z^2}{(1-z)^3}$$

(a) Find $Q(z)$ and d.

(b) Find a formula for $f(n)$.

(c) Find a formula for $f(n)$ if the denominator on the right hand side were instead $(1-z^2)^3$.

(D4) *Ehrhart series of the cross polytope.* Given a polytope $P = \operatorname{conv}\{v_1,\ldots,v_k\} \subset \mathbb{R}^d$ with the origin in its interior, define

$$\operatorname{Pyr}(P) = \operatorname{conv}\{(v_1,0),(v_2,0),\ldots,(v_k,0),e_{d+1}\} \subset \mathbb{R}^{d+1} \quad \text{and}$$
$$\operatorname{Bipyr}(P) = \operatorname{conv}\{(v_1,0),(v_2,0),\ldots,(v_k,0),e_{d+1},-e_{d+1}\} \subset \mathbb{R}^{d+1},$$

the *pyramid* and *bipyramid* over P, respectively (each lives in one dimension more than P).

(a) Let $P = [-1,1] \subset \mathbb{R}$. Draw P, $\operatorname{Pyr}(P)$ and $\operatorname{Bipyr}(P)$.

(b) Let $P = \operatorname{conv}\{(1,0),(-1,0),(0,1),(0,-1)\}$. Draw P, $\operatorname{Pyr}(P)$ and $\operatorname{Bipyr}(P)$.

(c) Find a general formula for $L_{\operatorname{Pyr}(P)}(3)$ in terms of $L_P(1)$, $L_P(2)$, and $L_P(3)$.

(d) Let $A(z) = \sum_{n\geq 0} a_n z^n$ and $B(z) = \sum_{n\geq 0} b_n z^n$, and suppose each $b_n = a_0 + \cdots + a_n$. Show that $(1-z)B(z) = A(z)$.

(e) Find a formula for $\operatorname{Ehr}_{\operatorname{Pyr}(P)}(z)$ in terms of $\operatorname{Ehr}_P(z)$ involving no sigma-sums.

(f) Find a sigma-sum free equation relating the Ehrhart series of P, $\operatorname{Pyr}(P)$, and $\operatorname{Bipyr}(P)$. Use this to obtain a formula for $\operatorname{Ehr}_{\operatorname{Bipyr}(P)}(z)$ in terms of $\operatorname{Ehr}_P(z)$.

(g) Find a formula for the Ehrhart series of the d-dimensional cross polytope

$$P_d = \operatorname{conv}\{e_1,-e_1,e_2,-e_2,\ldots,e_d,-e_d\} \subset \mathbb{R}^d.$$

(h) Why do we require that P have the origin in its interior?