Fall 2021, Math 596: Week 11 Problem Set
Möbius Functions and a Proof of Ehrhart's Theorem

Discussion problems. The problems below should be worked on in class.

(D1) *Triangulations and bookkeeping.*

(a) Draw a hexagon with vertices $A, B, C, \ldots$, and a triangulation $\mathcal{T}$ with no new vertices.

(b) Use $\mathcal{T}$ to express $\mathrm{Ehr}_P(z)$ in terms of the Ehrhart series of simplices. Your answer should **not** involve any rational expressions in z. For notational convenience, write $\mathrm{Ehr}_{ABC}(z)$ for the Ehrhart series of the simplex with vertices A, B, and C.

(c) Create a diagram that resembles a face lattice, but where the top element is P, the second row from the top is comprised of simplices in your triangulation, and the entries in each row below that are faces of one (or more) simplices in your triangulation. From top to bottom, there should be 1, 3, 9, and 6 entries in each row, respectively.

(d) For each entry F in your diagram, define $\mu(F)$ so that $\mu(P) = 1$ and for every $F \neq P$,

$$\sum_{G \supseteq F} \mu(G) = 0.$$

Use this to label each entry F in your diagram with its value $\mu(F)$.

(e) Compare the $\mu(F)$ values in part (d) with your expression in part (b).

(f) Given any polytope P and triangulation $\mathcal{T}$, conjecture a formula for $\mathrm{Ehr}_P(z)$ in terms of $\mathrm{Ehr}_F(z)$ for faces F of the simplices in $\mathcal{T}$.

(g) Repeat parts (a)-(d) for the following polytopes.

(i) A pentagon P, where the triangulation $\mathcal{T}$ has 5 simplices and one new vertex.
(ii) $P = \mathrm{conv}\{(1, 0, 0), (0, 1, 0), (-1, -1, 0), (0, 0, 1), (0, 0, -1)\}$.
(iii) A 3D-cube P (for this one, you are not required to draw the whole diagram).

(h) What do you notice about $\mu(F)$ when F lies on the boundary of P?

(i) Decide on the definitions of (i) a *simplicial cone*, and (ii) a *triangulation* of a cone C into simplicial cones. Write your definitions out **in full**.

(j) Locate a triangulation of

$$C = \mathrm{span}_{\geq 0}\{(1, 1, 0), (1, 0, 1), (0, 1, 1), (3, 1, 1), (1, 3, 1), (1, 1, 3)\},$$

and find an expression for σ_C in terms of the simplicial cones in your triangulation.

(D2) *Triangulation of the cross polytope.* In this problem, we will find the Ehrhart series of the cross polytope using triangulations.

(a) Let H denote a hexagon. Choose a triangulation of H, then locate a triangulation of $\mathrm{Pyr}(H)$ with the same number of simplices.

(b) Given a triangulation $\mathcal{T}$ of a polytope $P \subset \mathbb{R}^d$, locate a triangulation of $\mathrm{Pyr}(P)$ with the same number of simplices as $\mathcal{T}$ (this was previously a homework problem).

(c) Given a triangulation $\mathcal{T}$ of a polytope $P \subset \mathbb{R}^d$, locate a triangulation of $\mathrm{Bipyr}(P)$ with twice as many simplices as $\mathcal{T}$. State your conjecture formally.

(d) Locate a triangulation of the d-dimensional cross polytope using part (c). Use this to find the Ehrhart series when $d = 2, 3, 4$. Draw just enough of the diagram from (D1) to find the necessary coeffcients.