Fall 2021, Math 596: Week 15 Problem Set
Ehrhart-Macdonald Reciprocity

Discussion problems. The problems below should be worked on in class.

(D1) *Reading the lemma.* The goal of today's discussion is to prove the following, which will be central to the proof of Ehrhart-Macdonald Reciprocity on Thursday.

Lemma. *If $C \subset \mathbb{R}^d$ is a simplicial cone, $v \in \mathbb{R}^d$, and $v + C$ has no integer points on its boundary, then*

$$\sigma_{v+C}(z_1^{-1}, \ldots, z_d^{-1}) = (-1)^d \sigma_{-v+C}(z_1, \ldots, z_d).$$

The equality above requires **extreme** caution. Before proving it, we must closely examine its statement.

(a) We begin by revisiting the use of negative exponents. Carefully do the following.

(i) Write a few terms (including the constant term) in the power series expansions of

$$A(z_1) = \frac{1}{1 - z_1}, \qquad B(z_1) = \frac{1}{1 - z_1^{-1}}, \qquad \text{and} \qquad C(z_1) = \frac{1}{(1 - z_1)(1 - z_1^{-1})}$$

(ii) Verify that the series

$$A(z_1, z_2) = \frac{1}{(1 - z_1^{-1})(1 - z_2)} \qquad \text{and} \qquad B(z_1, z_2) = \frac{-z_1}{(1 - z_1)(1 - z_2)}$$

are equivalent as rational expressions. Then, determine whether the power series coefficients of $A(z_1, z_2)$ and $B(z_1, z_2)$ coincide.

(b) The moral of the examples in the previous part is as follows, which we shall refer to as *the safety rule of negative exponents.*

> There must exist a cone C such that **any** power series that comes into play has all of its exponents fall within some translation of C.

Identify where the safety rule was violated in each example from the previous part.

(c) With the safety rule in mind, examine the statement of the above lemma. Why is it clear we must proceed with **extreme** caution?

(d) Let $C = \text{span}_{\geq 0}\{(0, 1), (1, 0)\}$ and $v = (\frac{1}{2}, \frac{1}{2})$. Compute the series on the left and right hand sides of the statement in the lemma (for now, as separate entities).

(e) Find an interpretation of the left hand side that does not violate the safety rule when comparing it to the right hand side.

(f) Let $C = \text{span}_{\geq 0}\{(1, 1), (-1, 1)\}$ and $v = (0, \frac{3}{2})$. Verify that your interpretation in the previous part holds up, or adjust your interpretation so it does.

(D2) *Proving the lemma.* We are now ready to prove the above lemma.

(a) Let $C = \mathrm{span}_{\geq 0}\{(1,2),(3,1)\}$ and $v = (\frac{3}{2}, \frac{3}{2})$, and let Q denote the open parallelopiped of C. Draw

$$v + Q, \qquad -v + Q, \qquad -(-v + Q), \qquad \text{and} \qquad -(-v+Q) + (1,2) + (3,1).$$

(b) State and prove an equality, inspired by the previous part, for any vector v and the open parallelopiped Q of any simplicial cone C.

Hint: if $r_1, \ldots, r_k$ are the extremal ray vectors of C, then

$$Q = \{\lambda_1 r_1 + \cdots + \lambda_k r_k : \underline{\hspace{3cm}}\}$$

(c) Let $C = \mathrm{span}_{\geq 0}\{(1,2,3,4),(4,2,17,1),(3,11,5,3),(45,3,2,1)\}$ and $v = (\frac{1}{100}, \frac{11}{100}, \frac{37}{100}, \frac{19}{100})$. You may assume the hypotheses of the lemma are satisfied.

(i) Use the previous part to obtain an equation relating

$$\sigma_{-v+Q}(z_1, z_2, z_3, z_4) \qquad \text{and} \qquad \sigma_{v+Q}(z_1^{-1}, z_2^{-1}, z_3^{-1}, z_4^{-1}),$$

where Q is the open parallelopiped of C.

(ii) Verify the equality in the lemma.

(iii) Describe how one would generalize this example to prove the lemma.

(d) Where in your proof was the "has no integer points on its boundary" hypothesis used? Locate an example that demonstrates this hypothesis cannot be omitted.