Fall 2021, Math 596: Project Topics

The goal of each project is to learn about a topic not discussed in class. Throughout the semester, the following will be expected.

- Choose a topic. Please speak with me before making your decision, to ensure it is an appropriate level and so that we can narrow down a reasonable set of goals. You should choose a topic (and have it approved) no later than **Friday, November 12th**.
- Begin reading the agreed-upon background material. Plan to **meet at least twice** with me throughout the rest of the semester, to ensure that you are on track.
- Write (in LaTeX) a paper aimed at introducing your topic to fellow students, containing ample examples and explanations in addition to any theorems and proofs you give. Your writing should convey that you understand the intricacies of any proofs presented. Keep the following deadlines in mind as you proceed.
  - A rough draft of the paper will be due **Thursday, December 2nd** (one week before the last day of class). This will be peer reviewed by a fellow student.
  - The final paper will be due on **Tuesday, December 14th** (our "final exam" day).
- Graduate students only: give a 10-15 minute presentation introducing the main ideas of your topic. Presentations will take place during the final exam slot at the semester's end. You should keep in mind your target audience and available time when deciding what and how to present.

  Note: although the presentation is optional, everyone is encouraged to present and should strongly consider doing so. If there is sufficient interest, we can spend some time during the last week of classes on presentations as well.
- Your final grade on the project will be determined by the content, quality, and completeness of your final writeup (and, for graduate students, on the quality of the presentation).

Given below are several project ideas. Many of the listed sources contain more material than is necessary for the project, so be sure to meet with me so we can set reasonable project goals. I am also open to projects not listed here, but you must run them by me before making a decision. Don't be afraid to ask questions at any point during the project!

Polytopes

(1) *Inside-out polytopes.* A *graph* is a collection of dots (called *vertices*) and lines between them (called *edges*). A *proper coloring* of a graph is a way to color the vertices so that no two vertices on the same edge are the same color. The number of proper colorings on a graph G using n colors is known as the *chromatic polynomial* of G, and (as the name suggests) turns out to be a polynomial in n whose degree equals the number of vertices of G.

Chromatic polynomials of graphs can be studied using the Ehrhart functions of what are called *inside-out polytopes*, which are polytopes from which the intersections of certain hyperplanes through the middle have been removed.

Source: *Combinatorial reciprocity theorems* (M. Beck, R. Sanyal), Chapters 1 and 7.

(2) *Flow polytopes.* A *directed graph* is a collection of dots (called *vertices*) and arrow between them (called *edges*). A *flow* on a directed graph is a way to label each edge of G by an element of $\mathbb{Z}$ (or $\mathbb{Z}_n$) so that at each vertex v, the sum of the edges entering v equals the sum of the edges leaving v (i.e. the *in flow* at each vertex equals the *out flow*). The flows on a given directed graph can be counted using a certain family of polytopes, whose integer points each coincide with a flow on G.

Source: *Combinatorial reciprocity theorems* (M. Beck, R. Sanyal), Chapters 1 and 7.

(3) *Order polytopes.* A *partially ordered set* (or *poset* for short) is a set P and a reflexive, antisymmetric, and transitive *partial* ordering $\preceq$ (that is, some elements of P are incomparable under $\preceq$). One of the "standard" examples is a set P of positive integers (say, $\{1, \dots, 10\}$) under the partial order $a \preceq b$ if $a \mid b$. Indeed, divisibility is reflexive ($a \mid a$), antisymmetric ($a \mid b$ and $b \mid a$ implies $a = b$) and transitive ($a \mid b$ and $b \mid c$ implies $a \mid c$), but some integers are incomparable (e.g. $4 \nmid 6$ and $6 \nmid 4$).

An *order polytope* is a polytope $\mathcal{O}(P)$ constructed from a given finite poset P, and naturally encodes many properties of P. For example, the Ehrhart function of $\mathcal{O}(P)$ counts the number of *order preserving functions* on P.

Source: *Combinatorial reciprocity theorems* (M. Beck, R. Sanyal), Chapters 1, 2, and 6.

(4) *0/1 polytopes.* A 0/1 *polytope* is a polytope in which all vertex coordinates are either 0 or 1. Families of 0/1 polytopes can encode a wide variety of different combinatorics problems, and are often used to demonstrate "extremely bad behavior" that polytopes can exhibit.

Source: *Lectures on* 0/1 *polytopes* (G. Ziegler).

(5) *Transportation polytopes.* Frequently arising in optimization applications, transportation polytopes are a family of polytopes that play a central role in studying the traveling salesman problem (one of the classic computation problems in P vs. NP).

Source: *Combinatorics and Geometry of Transportation Polytopes: An Update* (J. De Loera, E. Kim).

(6) *Counting magic squares.* A *magic square* is an $n \times n$ grid of integers in which all row and column sums coincide. Counting the number of possible magic squares of a given size and row/column sum can be achieved using the Ehrhart functions of certain polytopes, wherein each integer point corresponds to a particular magic square.

Source: *Computing the continuous discretely* (M. Beck, S. Robins), Chapter 6.

(7) *Realizing associahdra.* We saw the associahedron $A_3 \subset \mathbb{R}^3$, and hinted at a larger family of polytopes A_n whose faces corresponded to (i) ways of drawing non-crossing diagonals in an $(n + 3)$-polygon, and (ii) ways of adding parenthesis to a non-associative product of $n + 1$ elements. Finding a "nice" choice of vertices for A_n turns out to be much harder than for the permutohedron, but the first successful attempt at doing so unlocked deep understanding about the structure of triangulations of polytopes.

Source: *Lectures on Polytopes* (G. Ziegler), Chapter 9.

(8) *The Shi arrangement and parking functions.* A collection of hyperplanes $H_1, \dots, H_k \subseteq \mathbb{R}^d$ can be viewed as cutting space into finitely many "regions" (i.e, the connected components of $\mathbb{R}^d \setminus (H_1 \cup \cdots H_k)$). The Shi arrangement is a particular arrangement of hyperplanes whose regions have especially nice combinatorial structure.

Source: *Parking functions, Shi arrangements, and mixed graphs* (M. Beck, A. Berrizbeitia, M. Dairyko, C. Rodriguez, A. Ruiz, S. Veeneman).

(9) *The Dehn-Sommerville relations.* The *face numbers* $f_0, f_1, \dots, f_d$ of a polytope P count the number of faces of P of dimension $0, 1, \dots, d$, respectively. If P is *simple* (meaning every vertex of P lies in exactly d edges, the smallest number geometrically possible), then the face numbers of P satisfy a collection of equations called the *Dehn-Sommerville relations.* These relations arise frequently in the study of Ehrhart functions of simple polytopes.

Source: *Computing the continuous discretely* (M. Beck, S. Robins), Chapter 5.

(10) *The Hirsch conjecture.* One of the most famous and longstanding open problems in the study of polytopes, the Hirsch conjecture (1957) proposed a bound on the possible distances between vertices in the 1-skeletons of polytopes. Since the first counterexample was discovered in 2012 (with a whopping dimension of 43), several smaller counterexamples have been located, though few (if any) are small enough to write down by hand. Although the conjecture is officially settled, many related questions still remain unanswered.

Source: *Who solved the Hirsch conjecture?* (G. Ziegler).

Generating functions

(11) *Multivariate quasipolynomials.* We will see that a function $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ is quasipolynomial if it is a polynomial with periodic coefficients (or, equivalently, that its generating function $\sum_{n \geq 0} f(n) z^n$ is rational in z with sufficiently nice numerator and denominator). What about multivariate functions $g : \mathbb{Z}_{\geq 0}^d \to \mathbb{R}$, with d integer inputs instead of just one? What does it mean for g to be a quasipolynomial? The answer to this question turns out to be somewhat more complicated than "a polynomial with periodic coefficients" but has surprising connections to geometry and cones in $\mathbb{Z}^d$.

Source: *Length functions determined by killing powers of several ideals in a local ring* (B. Fields), Chapter 2.

(12) *Generating functions as a counting tool.* Our use of generating functions in this class (to concisely express certain quasipolynomial functions) only scratches the surface of their utility. Generating functions are used heavily in combinatorics to give concise answers to counting questions when closed forms are difficult or impossible. The techniques for doing so yield an elegant high-level approach to combinatorics problems and some surprisingly slick proofs.

Source: *A walk through combinatorics* (3rd edition) (M. Bóna), Chapter 8.

Semigroups

(13) *The Kunz polyhdedron.* A numerical semigroup is a subset $S \subseteq \mathbb{Z}_{\geq 0}$ that is closed under addition whose complement in $\mathbb{Z}_{\geq 0}$ is finite. For each $m \geq 3$, there is a polyhedron P_m, shaped like a pointed cone translated slightly from the origin, whose integer points each correspond to a numerical semigroup containing m. The faces of these polyhedra naturally divide the collection of all numerical semigroups containing m (an infinite collection) into finitely many "classes" with similar algebraic structure.

Source: *Wilf's conjecture in fixed multiplicity* (W. Bruns, P. García-Sánchez, C. O'Neill, D. Wilbourne).

(14) *Counting oversemigroups.* A numerical semigroup is a subset $S \subseteq \mathbb{Z}_{\geq 0}$ that is closed under addition whose complement in $\mathbb{Z}_{\geq 0}$ is finite. A numerical semigroup T is an *oversemigroup* of a numerical semigroup S if $T \supset S$. Every numerical semigroup has only finitely many oversemigroups since there are only finitely many gaps to "fill in". Recently, Ehrhart functions were used in counting the number of oversemigroups by constructing certain polytopes in which each integer point corresponds to an oversemigroup of the given semigroup.

Source: *On the number of numerical semigroups containing two coprime integers* (M. Hellus, R. Waldi).

Advanced algebra topics

(15) *Hilbert functions.* Hilbert functions, which arise in the study of polynomial ideals, provide a natural bridge between commutative algebra and combinatorics, and are closely related to much of the content we will see in this course. Hilbert's theorem states that certain Hilbert

functions are eventually quasipolynomial (this is in many ways a "3rd viewpoint" alongside polytopes and generating functions). Among other things, Hilbert's theorem can be used to give an algebraic proof of Ehrhart's theorem for integral/rational polytopes.

Source: *Combinatorial commutative algebra* (E. Miller, B. Sturmfels), Chapter 8.

Note: This project requires at least one semester of abstract algebra, but preferably two. At the very least, familiarity with ideals, polynomial rings, and quotient rings is essential.

(16) *Binomial ideals.* An ideal I in the polynomial ring $\mathbb{Q}[x_1, \ldots, x_d]$ is a *binomial ideal* if it can be generated by differences of monomials. Binomial ideals provide a natural bridge between commutative algebra, semigroups, and polyhedral cones, and are hiding in the background throughout much of the content in this course (in some ways, this is a "4th viewpoint" alongside Hilbert functions, polytopes, and generating functions). For instance, while we may view subtraction in the numerator of a rational generating function as "inclusion-exclusion", from the semigroups viewpoint, these differences encode relations between semigroup generators.

Source: *Combinatorial commutative algebra* (E. Miller, B. Sturmfels), Chapter 7.

Note: This project requires at least one semester of abstract algebra, but preferably two. At the very least, familiarity with ideals, polynomial rings, and quotient rings is essential.

Applications

(17) *Voting theory.* One of many applications of Ehrhart theory is in the analysis and comparison of certain voting methods, wherein the coordinates of each integer point represent the number of votes some candidate received, and the possible voting totals, as well as the possible election outcomes, can be viewed as linear inequalities.

Source: *On Ehrhart polynomials and probability calculations in voting theory* (D. Lepelley, A. Louichi, H. Smaoui).

(18) *Simplex method.* Central to linear programming, operations research, and combinatorial optimization, the simplex method is an algorithm for finding an optimal solution for a system of linear inequalities. It's generality and efficiency enable its use in countless applications.

Source: *Introduction to the simplex method* (T. Hu and A. Kahng).