Fall 2021, Math 620: Week 2 Problem Set
Due: Thursday, September 9th, 2021
Permutation Groups

Discussion problems. The problems below should be completed in class.

(D1) *Working with permutations.* Consider the following permutations.

$$\sigma = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 3\,5\,2\,4\,1 \end{pmatrix} \quad \tau = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ 4\,1\,6\,3\,8\,2\,5\,7 \end{pmatrix} \quad \alpha = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ 5\,2\,8\,6\,1\,4\,7\,3 \end{pmatrix}$$

- (a) Write each of the above permutations as a product of disjoint cycles.
- (b) Find the order of each element above. Hint: do the disjoint cycles help?
- (c) Write each of the above permutations as a product of 2-cycles in two different ways.
- (d) Determine which of the above permutations are even, and which are odd.
- (e) Is it possible to write σ or τ as a product of disjoint 2-cycles?
- (f) Write σ^{-1}, τ^{-1}, and α^{-1} as a products of disjoint cycles, and as products of 2-cycles. Hint: you have already written σ, τ, and α in these forms!
- (g) Determine whether σ^{91} is even or odd.

(D2) *The alternating group.* Let $A_n \subset S_n$ denote the set of even permutations of n.

- (a) Find all elements of A_3.
- (b) Prove that A_n is a subgroup of S_n.
- (c) Find a formula for $|A_n|$ in terms of n.

(D3) *Group elements as permutations.* The goal of this problem is to develop intuition behind the following theorem.

Theorem. *Every finite group G with $|G| = n$ is isomorphic to a subgroup of S_n.*

- (a) The goal of the first few parts is to identify a subgroup of S_6 isomorphic to D_3. Choose a labeling of the elements of D_3 as $a_1, a_2, \ldots, a_6$ (you may do this in any way you wish).
- (b) Let $r \in D_3$ denote clockwise rotation by 120°. Define a permutation $\sigma \in S_6$ given by $\sigma(i) = k$ whenever $a_k = ra_i$. Verify that σ is indeed a permutation by writing it in permutation notation.
- (c) Repeat the previous part for each $a \in D_3$ (that is, define a permutation $\sigma_a \in S_6$ given by $\sigma(i) = k$ where $a_k = aa_i$). Write down all 6 resulting permutations (you may want to "divide and conquer" amongst your groupmates to save time!).
- (d) Verify in three examples that for any $a, b \in D_3$, the permutation corresponding to ab equals the product of the permutations corresponding to a and b.
- (e) Using the ideas above, find a subgroup of S_4 isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.
- (f) Using the ideas above, find a subgroup of S_n isomorphic to $\mathbb{Z}_n$.
- (g) Fix a group G, with elements labeled $a_1, \ldots, a_n$, and an element $a \in G$. Prove the function σ_a defined by $\sigma(i) = k$ whenever $a_k = ra_i$ is indeed a permutation in S_n.
- (h) Prove that the function $\varphi : G \to S_n$ given by $a \mapsto \sigma_a$ is an isomorphism onto its image.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) It turns out $GL_2(\mathbb{Z}_2)$ is isomophic to either S_3 or $\mathbb{Z}_6$. Determine which.

(H2) Identify an element of S_9 of order 20.

(H3) Prove S_n is isomorphic to a subgroup of A_{n+2}.

(H4) Determine whether each of the following statements is true or false. Prove your assertions.

   (a) The group $(\mathbb{Q}, +)$ is cyclic.
   (b) For each $n \geq 3$, every permutation in S_n can be written as a product of 3-cycles.
   (c) For each $n \geq 3$, every permutation in S_n is a product of at most $n-1$ transpositions.
   (d) For each $n \geq 3$, every permutation in S_n is a product of adjacent transpositions.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Locate a generating set for S_n consisting of only 2 generators.