Fall 2021, Math 620: Week 3 Problem Set
Due: Thursday, September 16th, 2021
Cosets and Quotient Groups

Discussion problems. The problems below should be completed in class.

(D1) *Geometry and cosets.* This problem concerns the subgroup $H = \langle (1,3), (3,1) \rangle \subset \mathbb{Z}^2$.

(a) Draw a grid of points from $(-2,-2)$ to $(8,8)$. Draw the axes as well.

(b) Circle the elements of H on the grid **in black**.

(c) Choose an element $a \notin H$, and circle the points in $a + H$ in a color **other than black**.

(d) Choose an element $a' \notin H \cup (a + H)$, and circle the points in $a' + H$ **in a third color**.

(e) Draw lines connecting the origin to the generators of H, as well as from the generators of H to their sum (this should form a parallelogram). Does this indicate a choice of "natural" representatives?

(f) Determine what more familiar group G/H is isomorphic to. Hint: determine $|G/H|$, and find the orders of some of the elements.

(D2) *Arithmetic in quotient groups.* Consider the unit interval as an additive group, viewed as the quotient group $\mathbb{R}/\mathbb{Z}$.

(a) Explain why every element of $\mathbb{R}/\mathbb{Z}$ has a unique representative in $[0, 1)$.

(b) Show that $\mathbb{R}/\mathbb{Z}$ has elements of every possible order.

(c) Does $\mathbb{Q}/\mathbb{Z}$ have elements of every possible order?

(d) Does $\mathbb{Q}/\mathbb{Z}$ have infinitely many elements of a particular order?

(D3) *Subgroup correspondence.* The goal of this problem is to discover the "correspondence theorem" for subgroups of quotient groups.

(a) Find all 4 subgroups of $\mathbb{Z}_6$, and list the elements of each. Draw the subgroups in a diamond shape, with $\mathbb{Z}_4$ at the top and $\{0\}$ at the bottom, and lines between subgroups indicating set containment (this is called the *subgroup lattice* of $\mathbb{Z}_6$).

(b) Draw the subgroup lattice of $\mathbb{Z}_{24}$. Write each subgroup in the form $\langle n \rangle$.

(c) Do the same for the subgroup lattice of $\mathbb{Z}_{72}$.

(d) Locate a subgroup $H \subset \mathbb{Z}_{72}$ so that $\mathbb{Z}_{72}/H \cong \mathbb{Z}_{24}$. Draw a circle **in a different color** around the subgroups of $\mathbb{Z}_{72}$ containing H. What do you notice about the **shape** of these subgroups and the **shape** of the subgroup lattice of $\mathbb{Z}_{24}$?

(e) Find all 10 subgroups of the group D_4, and draw the subgroup lattice (draw big, and write each subgroup as a set of elements). Draw a big circle around the collection of subgroups containing $H = \{e, r^2\}$ (including H and G).

(f) Choose a coset of H, and mark all listed elements **in a second color**.

(g) Choose another coset of H, and do the same **in a third color**.

(h) Does any subgroup of D_4 containing H appear to contain any "partial" cosets of H?

(i) It turns out $D_4/H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Draw the subgroup lattice of $\mathbb{Z}_2 \times \mathbb{Z}_2$. What do you notice about its shape compared to the shape of the subgroups of D_4 containing H?

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Suppose G is a group and $H \subseteq G$ is a subgroup. Prove **from first principles** (i.e., without using any theorems) that H is normal if and only if $aha^{-1} \in H$ for all $h \in H$, $a \in G$.

(H2) Fix a group G. Define the *center* of G as the set

$$C = \{c \in G : ca = ac \text{ for all } a \in G\}$$

of elements that commute with every element of G.

- (a) Prove C is a normal subgroup of G.
  Hint: use Problem (H1).
- (b) Prove or disprove: G/C has trivial center.

(H3) Fix a group $(G, \cdot)$ and a normal subgroup $H \subseteq G$. In what follows, consider the map

$$\begin{aligned} \varphi : G &\longrightarrow G/H \\ a &\longmapsto [a]_H. \end{aligned}$$

- (a) Prove if $K \subseteq G/H$ is a subgroup, then $H' = \varphi^{-1}(K)$ is a subgroup of G containing H.
  Note: the elements of K are **cosets** of H.
- (b) Prove if $H' \subseteq G$ is a subgroup with $H \subseteq H'$, then $K = \varphi(H')$ is a subgroup of G/H.
- (c) Conclude that there is a bijective correspondence

$$\{\text{subgroups } H' \subseteq G \text{ containing } H\} \longleftrightarrow \{\text{subgroups } K \subseteq G/H\}$$

  that preserves containment.
  Note: there is no work to be done for this part.

(H4) The following questions pertain to the correspondence theorem from Problem (H3). Prove any assertions you make.

- (a) If a subgroup of G/H is normal, is its corresponding subgroup of G normal? What about the converse of this statement?
- (b) If a subgroup of G/H is cyclic, is its corresponding subgroup of G cyclic? What about the converse of this statement?

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Suppose p is prime. Find all groups G (up to isomorphism) with $|G| = p^2$.