Fall 2021, Math 620: Week 4 Problem Set
Due: Thursday, September 23rd, 2021
Isomorphism Theorems

Discussion problems. The problems below should be worked on in class.

(D1) *Homomorphisms.* Fix a group homomorphism $\varphi : G \to G'$.

(a) Prove $\varphi(e) = e$. Where does each "e" live?

(b) Prove for all $a \in G$, $\varphi(a^{-1}) = (\varphi(a))^{-1}$.

(c) Prove φ is injective if and only if $\ker(\varphi)$ is trivial.

(d) Prove $\mathrm{Im}(\varphi)$ is a subgroup of G'.

(e) Prove that $G/\ker(\varphi) \to \mathrm{Im}(\varphi)$ given by $[a] \mapsto \varphi(a)$ is an isomorphism.
Hint: we have already shown the map is well-defined.

(D2) *The first isomorphism theorem.* Prove the following using the first isomorphism theorem.

(a) $G/\{e\} \cong G$ and $G/G \cong \{e\}$ for any group G.

(b) $S_n/A_n \cong \mathbb{Z}_2$.

(c) $GL_n(\mathbb{R})/SL_n(\mathbb{R}) \cong (\mathbb{R} \setminus \{0\}, \cdot)$.

(d) $GL_n(\mathbb{R})/D \cong SL_n(\mathbb{R})$, where $D = \{\lambda I_n : \lambda \in \mathbb{R} \setminus \{0\}\}$ is the set of nonzero scalar multiples of the identity matrix.

(D3) *The third isomorphism theorem.* The goal of this problem is to prove the following theorem. Tip: due to the large number of quotients in this problem, **use the bracket notation** for all quotient group elements (e.g., $[a]_H$).

Theorem. *If $(G, \cdot)$ is a group and $K, H \lhd G$ with $K \subset H$, then $(G/K)/(H/K) \cong G/H$.*

(a) Let $G = \mathbb{Z}_{24}$, $H = \{[3k]_{24} : k \in \mathbb{Z}\}$, and $K = \{[12k]_{24} : k \in \mathbb{Z}\}$. Verify the above theorem holds in this case.

(b) Explain why H/K is a subset of G/K. Use the word "coset" in your explanation.

(c) Complete each of the following **without** using any existing isomorphism theorems.

(i) Prove that H/K is a subgroup G/K.

(ii) Propose a map $\varphi : (G/K)/(H/K) \to G/H$ to serve as a starting place for the isomorphism. Why is your map a natural choice?

(iii) Prove your map φ is well defined.

(iv) Verify your map φ is a homomorphism.

(v) Complete the proof by proving φ is a bijection.

(d) Can the above proof be shortened using the first isomorphism theorem? In particular, can we define a homomorphism $\phi : G/K \to G/H$ whose kernel equals H/K?

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) **Using the first isomorphism theorem**, prove that for any groups G and G',

$$G \times G'/(G \times \{e\}) \cong G'.$$

(H2) An *automorphism* of a group G is an isomorphism $\varphi : G \to G$. Denote the set of automorphisms of G by $\mathrm{Aut}(G)$.

(a) Prove that $\mathrm{Aut}(G)$ is a group under function composition.

(b) Determine which familiar group $\mathrm{Aut}(\mathbb{Z})$ is isomorphic to.

(H3) Suppose G is a group. Given $a \in G$, define $f_a : G \to G$ by $f_a(x) = axa^{-1}$.

(a) Prove f_a is an automorphism (these are known as *inner automorphisms*).

(b) Let $\mathrm{Inn}(G) = \{f_a : a \in G\} \subseteq \mathrm{Aut}(G)$. Prove $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

(c) Let $\varphi : G \to \mathrm{Inn}(G)$ denote the map $a \mapsto f_a$. Characterize the elements of G in $\ker(\varphi)$.

(d) Characterize which groups G have a unique inner automorphism.

(H4) Determine whether each of the following statements is true or false. Prove your assertions.

(a) If G is a group and $H, K \lhd G$ with $K \subset H$, then $G/H \times H/K \cong G/K$.

(b) If G, G' are groups and $H \lhd G$, $H' \lhd G'$, then $(G \times G')/(H \times H') \cong (G/H) \times (G'/H')$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Suppose G is a group and $H, K \lhd G$ with $HK = G$. Determine under what condition(s) involving H and K we have $G \cong G/H \times G/K$.