Fall 2021, Math 620: Week 5 Problem Set
Due: Thursday, September 30th, 2021
Relations and Group Presentations

Discussion problems. The problems below should be completed in class.

(D1) *Free groups and relations.*

(a) Let $F_1 = \langle a \rangle$ denote the free group on 1 generator. Locate an isomorphism $F_1 \to \mathbb{Z}$. (You do not have to prove it is an isomorphism.)

(b) Draw the Cayley graph of the group $G = \langle r, f \mid f^2, rfrf \rangle$ (this is identical to the presentation of D_n without the r^n relation). Is G finite or infinite?

(c) Let $G = \langle a, b \mid a^2, b^2, aba^{-1}b^{-1} \rangle$. Determine what more familiar group G is.

(D2) *Generating normal subgroups.* Let $F_2 = \langle a, b \rangle$ denote the free group on 2 generators.

(a) Find the smallest subgroup of D_4 containing f (i.e., the subgroup generated by f).

(b) Find the smallest **normal** subgroup of D_4 containing f. Does this differ from part (a)?

(c) One way to obtain the smallest subgroup of a group G containing given elements g and h is to take all possible products of g and h and their inverses. What additional steps must we take to obtain the smallest **normal** subgroup containing g and h?

(d) Use the previous part to identify the smallest subgroup of D_8 containing f.

(e) Let $H = \langle a \rangle$ denote the smallest **normal** subgroup of F_2 containing a.

(i) Determine which familiar group $F_2/H = \langle a, b \mid a \rangle$ is isomorphic to.

(ii) Demonstrate $a^2b^{-2}a^{17}b^5a^3b^{-3}a^{-2} \in H$ using your answer to part (c) above.

(iii) Determine $a^2b^{-2}a^{17}b^5a^3 \notin H$ using your answer to part (i) above.

(f) From here on, we must pay **close attention** when using brackets $\langle\ \rangle$ to define a subgroup that we specify **in words** whether we mean the **plain** subgroup with those generators or the **normal** subgroup with those generators.

Note: there is nothing to do for this part. Internalize the statement, then continue on.

(D3) *Group presentations.* Let $F_2 = \langle a, b \rangle$ denote the free group on 2 generators.

(a) For each of the following **normal** subgroups H_1 and H_2, prove $H_1 = H_2$ by representing each generator of H_1 as a product of conjugates of generators of H_2, and visa versa.

(i) $H_1 = \langle a^3b^2 \rangle$, $H_2 = \langle ba^3b \rangle$.

(ii) $H_1 = \langle aba^{-1}b^{-1}, a^2b^3ab^4 \rangle$, $H_2 = \langle aba^{-1}b^{-1}, a^3b^7 \rangle$.

(iii) $H_1 = \langle a^4ba^{-3}b^{-2}, a^3b^2a^{-2}b^{-3}, a^2b^3a^{-1}b^{-4} \rangle$,
$H_2 = \langle a^4ba^{-3}b^{-2}, a^3b^2a^{-2}b^{-3}, a^4ba^{-1}b^{-4} \rangle$.

(b) Consider the **normal** subgroup $H = \langle a^4, b^4, a^2b^{-2}, abab^{-1} \rangle$, and let $G = F_2/H$. Find $|G|$, and determine whether G is Abelian.

(c) Choose $\sigma, \tau \in S_3$ that generate S_3. Locate generators for a normal subgroup $H \subset F_2$ so that $F_2/H \cong S_3$ with $a \mapsto \sigma$ and $b \mapsto \tau$.

(d) Let $F_3 = \langle x, y, z \rangle$ denote the free group on 3 generators. Locate a (not necessarily normal) subgroup of F_2 that is isomorphic to F_3.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) The following theorem is known as the Fundamental Theorem of Finite Abeilan Groups.

Theorem. *Every finite Abelian group is isomorphic to a direct product of cyclic groups.*

Using **this statement of the Fundamental Theorem** (i.e., no specializations found in a textbook or other sources), find all Abelian groups of order 24 up to isomorphism. Ensure your list does not have any duplicates!

(H2) Let $G = \langle a, b \mid aba^{-1}b^{-2}, bab^{-1}a^{-2} \rangle$. Determine $|G|$.

(H3) Identify which "familiar" group each of the following is isomorphic to.

(a) $G = \langle a, b \mid a^4, b^2, aba^{-1}b^{-1} \rangle$

(b) $G = \langle a, b, c \mid a^4, b^2, c^2, abc, acac \rangle$

(H4) Determine whether each of the following statements is true or false. Prove your assertions.

(a) If $F_2 = \langle a, b \rangle$ is the free group on 2 generators, then the **subgroup** $H = \langle ab, ab^2 \rangle \subseteq F_2$ and **normal subgroup** $H' = \langle ab, ab^2 \rangle \subseteq F_2$ are identical subsets of F_2.

(b) The group $(\mathbb{Q}, +)$ is a quotient of the free group on 2 generators.

(c) Up to isomorphism, the only non-Abelian group of order 8 is D_4.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Determine which familiar group is isomorphic to $G = \langle a, b \mid a^4, b^2, (ab)^3 \rangle$.