Fall 2021, Math 620: Week 7 Problem Set
Due: Thursday, October 14th, 2021
Ideals and Quotient Rings

Discussion problems. The problems below should be worked on in class.

(D1) *Determining ring homomorphisms with minimal effort.*

(a) Suppose $\varphi : \mathbb{Z}[x] \to R$ is a ring homomorphism, and that $\varphi(1) = 1_R$ and $\varphi(x) = a$. Determine $\varphi(x^3 + 2x^2 + 17x - 4)$ in terms of a.

(b) Suppose $\varphi : \mathbb{Q}[x] \to R$ is a ring homomorphism, and that $\varphi(1) = 1_R$ and $\varphi(x) = a$. Determine $\varphi(\frac{5}{2})$ and $\varphi(3x^2 - \frac{1}{3}x + \frac{17}{52})$ in terms of a.

(c) Does there exist a nontrivial ring homomophism $\varphi : \mathbb{Q} \to \mathbb{Z}$?

(D2) *Polynomial ring quotients.*

(a) Let $I = \langle x^2 + 5x, x + 2\rangle = \mathbb{Q}[x]$. Demonstrate that $I = \mathbb{Q}[x]$.
Hint: recall that $I = \{(x^2 + 5x)f(x) + (x + 2)g(x) : f, g \in \mathbb{Q}[x]\}$

(b) Locate a single element that generates the ideal $I = \langle x^2 - 2x, x^3 + 5x^2\rangle \subseteq \mathbb{Q}[x]$.

(c) Consider the quotient ring $R = \mathbb{R}[x]/\langle x^2 + 1\rangle$.

(i) Argue that every element can be written in the form $[ax + b]$ for some $a, b \in \mathbb{R}$.

(ii) What happens if you square the element $[x]$? How is this similar to $i \in \mathbb{C}$?

(iii) Use the first isomorphism theorem to prove $\mathbb{R}[x]/\langle x^2 + 1\rangle \cong \mathbb{C}$.

(d) Let $\mathbb{Q}[\sqrt{2}] = \{f(\sqrt{2}) : f(x) \in \mathbb{Q}[x]\}$. What is the simplest way we can write a general element of $\mathbb{Q}[\sqrt{2}]$? Write a general multiplication rule for $\mathbb{Q}[\sqrt{2}]$.

(e) Identify $\mathbb{Q}[\sqrt{2}]$ as a quotient of the polynomial ring $\mathbb{Q}[x]$. Use the first isomorphism theorem to prove your claim.

(f) Determine whether $\mathbb{R}[x]/\langle x^2 - 4\rangle$ is an integral domain, a field, or neither.

(g) Suppose R is a ring such that some $t \in R \setminus \{1, -1\}$ has $t^2 = 1$. Prove R is not a field. Find an example of such a ring.

(D3) *Prime and maximal ideals.* Fix a commutative ring R and an ideal I. We say I is *prime* if R/I is an integral domain, and I is *maximal* if R/I is a field.

(a) Suppose R is a field. Classify all ideals of R.

(b) Suppose I is maximal, and fix an ideal J with $I \subsetneq J \subseteq R$. What can be said about J?

(c) State and prove an equivalent condition for I to be maximal that avoids quotient rings.

(d) Suppose I is prime, and fix ring elements $a, b \notin I$. Under what conditions is $ab \in I$?

(e) State and prove an equivalent condition for I to be prime that avoids quotient rings.

(f) Let $R = \mathbb{Z}[x]$ and $I = \langle 2, x\rangle$. Prove I is maximal by identifying R/I.

(g) Identify 3 distinct maximal ideals in $\mathbb{Q}[x, y]$.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Consider the rings $R_1 = \mathbb{Q}[x]/\langle x^2 - 1\rangle$ and $R_2 = \mathbb{Q}[x]/\langle x^2 + 1\rangle$.

- (a) Are R_1 and R_2 isomorphic as additive groups?
- (b) Are R_1 and R_2 isomorphic as rings?

(H2) Fix an ideal $I \subset R$. We say I is *prime* if R/I is an integral domain and I is *maximal* if R/I is a field. Consulting your favorite abstract algebra textbook (the lecture schedule on the course webpage lists the relevant sections in both Dummit & Foote and Hungerford), read and internalize the proofs that

- (a) I is prime if and only if $ab \in I$ implies $a \in I$ or $b \in I$, and
- (b) I is maximal if and only if $I \subsetneq R$ and there is no ideal J with $I \subsetneq J \subsetneq R$.

Once you have completed this, it suffices to write "DONE" as your answer.

(H3) Consider the ring

$$\mathbb{Q}[\![x]\!] = \{a_0 + a_1x + a_2x^2 + \cdots : a_i \in \mathbb{Q}\}$$

of *formal power series* with coefficients in $\mathbb{Q}$. Here, addition is term-by-term, and multiplication is given by distribution using the following formula.

$$(a_0 + a_1x + a_2x^2 + \cdots)(b_0 + b_1x + \cdots) = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \cdots$$

Notice that on the right side above, each coefficient involves only a **finite** sum! This way we avoid any questions of convergence.

- (a) Determine which elements of $\mathbb{Q}[\![x]\!]$ are units.
- (b) Find all maximal ideals of $\mathbb{Q}[\![x]\!]$.

(H4) Determine whether each of the following statements is true or false. Prove your assertions.

- (a) Every nonzero element of $R = \mathbb{Q}[x]/\langle x^2\rangle$ is either a unit or a zero-divisor.
- (b) The characteristic of any field is either 0 or prime.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Prove or disprove: any integral domain R with finitely many elements is a field.