Fall 2021, Math 620: Week 8 Problem Set
Due: Thursday, October 21st, 2021
Rings of Fractions and Localization

Discussion problems. The problems below should be worked on in class.

(D1) *Constructing the rationals from the integers.* Define an equivalence relation $\sim$ on $\mathbb{Z} \times \mathbb{Z}_{\neq 0}$ by $(a, b) \sim (a', b')$ when $ab' = a'b$. Let Q denote the set of equivalence classes of $\sim$. Define operations $\oplus$ and $\odot$ on Q so that

$$[(a, b)] \oplus [(c, d)] = [(ad + bc, bd)] \qquad \text{and} \qquad [(a, b)] \odot [(c, d)] = [(ac, bd)]$$

for all $[(a, b)], [(c, d)] \in Q$. (Intiutively, $[(a, b)] \in Q$ represents $a/b \in \mathbb{Q}$.)

(a) Prove addition is well-defined in Q, that is, if $[(a, b)] = [(a', b')]$ and $[(c, d)] = [(c', d')]$, then $[(a, b)] \oplus [(c, d)] = [(a', b')] \oplus [(c', d')]$. Do the same for multiplication.
(b) Prove addition in Q is commutative and associative.
(c) Prove multiplication in Q is commutative and associative.
(d) Prove distributivity holds in Q.
(e) Prove every element of Q has an additive inverse, and that every nonzero element has a multiplicative inverse.

(D2) *Constructing the integers from the naturals.* Define an equivalence relation $\sim$ on $\mathbb{N} \times \mathbb{N}$ by $(a, b) \sim (a', b')$ when $a + b' = a' + b$. Let Z denote the set of equivalence classes of $\sim$. Define operations $\oplus$ and $\odot$ on Z so that

$$[(a, b)] \oplus [(c, d)] = [(a + c, b + d)] \qquad \text{and} \qquad [(a, b)] \odot [(c, d)] = [(ac + bd, ad + bc)]$$

for all $[(a, b)], [(c, d)] \in Z$. (Intiutively, $[(a, b)] \in Z$ represents $a - b \in \mathbb{Z}$.)

(a) Prove addition is well-defined in Z, that is, if $[(a, b)] = [(a', b')]$ and $[(c, d)] = [(c', d')]$, then $[(a, b)] \oplus [(c, d)] = [(a', b')] \oplus [(c', d')]$. Do the same for multiplication.
(b) Prove addition in Z is commutative and associative.
(c) Prove multiplication in Z is commutative and associative.
(d) Prove distributivity holds in Z.
(e) Prove every element of Z has an additive inverse.

(D3) *Constructing the natural numbers.* Let N be a set constructed in the following way:

- there is a distinguished element $0 \in N$;
- every $a \in N$ has a *successor* $a^+ \in N$; and
- 0 is not the successor of any element.

Define the operations $\oplus$ and $\odot$ on N so that

$$a \oplus 0 = a \qquad a \oplus b^+ = (a \oplus b)^+ \qquad a \odot 0 = 0 \qquad \text{and} \qquad a \odot b^+ = (a \odot b) \oplus a$$

for all $a, b \in N$. Every proof in this part will use "induction": if a given statement holds for $0 \in N$ and it holds for a^+ whenever it holds for a, then it holds for every element of N.

(a) For convenience, let $1 = 0^+$, $2 = 1^+$, and $3 = 2^+$. Find $a \oplus b$ for every $a, b \in \{0, 1, 2, 3\}$.
(b) Prove that $a^+ \oplus b = (a \oplus b)^+$ for every $a, b \in N$.
(c) Prove that addition is commutative and associative in N.
(d) Prove that $a^+ \odot b = (a \odot b) \oplus b$ for every $a, b \in N$.
(e) Prove that multiplication is commutative and associative in N.
(f) Prove that distributivity holds in N.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) In each of the following, you may use "free of charge" (i.e., without proof) any prior parts of the same discussion problem, but **not** any from previous discussion problems.

(a) Write up a proof for any one part of Problem (D1) you choose.

(b) Write up a proof for any one part of Problem (D2) you choose (preferably involving different adjective(s) than your choice from (D1)).

(c) Write up a proof for any one part of Problem (D3)$\setminus\{$(a)$\}$ you choose (preferably involving different adjective(s) than your choices from (D1) and (D2)).

(H2) Fix a field F. Prove that F has a subring isomorphic to $\mathbb{Q}$ if and only if $\operatorname{char} F = 0$.

(H3) Fix a ring R and a prime ideal $P \subset R$, and let $D = R \setminus P$.

(a) Prove that D is a multiplicative set.

(b) Prove that the ring $R_P = D^{-1}R$ has a unique maximal ideal (the ring R_P is called the *localization of* R *at* P and plays an important role in algebraic geometry).

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Given a ring R and a prime ideal P, classify the prime ideals of R_P in terms of the prime ideals of R.