Fall 2021, Math 620: Week 11 Problem Set
Due: Thursday, November 18th, 2021
Field Extensions

Discussion problems. The problems below should be worked on in class.

(D1) *Splitting fields.*

(a) Find the minimal polynomials of $\sqrt{2}$, $3\sqrt{2}+4$, and $\sqrt[3]{2}+1$ over $\mathbb{Q}$.

(b) Find the minimal polynomial of $\sqrt{2}+\sqrt{3}$ over $\mathbb{Q}$.

(c) Argue that $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2},\sqrt{3})$.

(d) Does part (c) hold if 2 and 3 are replaced with any distinct positive integers m, n?

(e) Find the splitting field of $(x^2-5)(x^2-7)$.

(f) Find the splitting field of x^4-4 over $\mathbb{Q}$.

(g) Find the splitting field of x^d-1 over $\mathbb{Q}$ for $d = 2$, 3, 4, 5, and 6.

(h) Conjecture a formula in $d \ge 2$ for the degree of the splitting field of x^d-1 over $\mathbb{Q}$.

(D2) *Algebraic closures.* Fix a field F. An *algebraic closure* of F is an algebraically closed field $\overline{F}$ that is algebraic over F (note this is stronger than simply being algebraically closed and containing F). The goal of this problem is to prove the first part of the following theorem, using (without proof) that every field is contained in **some** algebraically closed field.

Theorem. *There exists an algebraically closed field $\overline{F}$ that is algebraic over F. Moreover, $\overline{F}$ is unique up to isomorphism.*

(a) Find the splitting field of $f(x) = x^2+\sqrt{2}x+1$ over $\mathbb{Q}(\sqrt{2})$. Find the minimal polynomial of each root of $f(x)$ over $\mathbb{Q}$.

(b) Suppose α is algebraic over F and β is algebraic over $F(\alpha)$. Use the first isomorphism theorem to prove that β is the root of some irreducible polynomial in $F[x]$.

(c) Suppose $F \subset C$ for some algebraically closed field C, and let $F' \subset C$ denote the set of elements that are algebraic over F. Prove that F' is a field containing F.

(d) Prove that the field F' in the previous part is algebraically closed.

(e) Conclude that F' is an algebraic closure of F.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Find the minimal polynomial of $\sqrt{3} + \sqrt[3]{2}$ over $\mathbb{Q}$.

(H2) Find the splitting field of $x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ over $\mathbb{Z}_3$.

(H3) Fix a prime p. In this problem, we will construct the algebraic closure of $\mathbb{Z}_p$. Hint: you may find The Key Lemma useful frequently!

- (a) Fix $t \geq 1$. Prove $\mathbb{F}_{p^t}$ has a subfield isomorphic to $\mathbb{F}_{p^r}$ if and only if $r \mid t$.
- (b) Prove that if $r \mid t$, then there is a **unique** subfield of $\mathbb{F}_{p^t}$ isomorphic to $\mathbb{F}_{p^r}$.
  In light of this, in what follows, when $r \mid t$, it is natural to write $\mathbb{F}_{p^r} \subset \mathbb{F}_{p^t}$, identifying $\mathbb{F}_{p^r}$ with the subfield of $\mathbb{F}_{p^t}$ it is isomorphic to.
- (c) Let $F = \bigcup_{t \geq 1} \mathbb{F}_{p^t}$. Prove that F is a field.
- (d) Prove that F is algebraically closed.
- (e) Prove that $F = \overline{\mathbb{Z}}_p$, i.e., there is no algebraically closed field F' with $\mathbb{F}_q \subseteq F' \subsetneq F$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Locate a field F that has cardinality strictly larger than $\mathbb{R}$.