Fall 2021, Math 620: Week 13 Problem Set
Due: Tuesday, November 23rd, 2021
Introduction to Modules

Discussion problems. The problems below should be worked on in class.

(D1) *Modules.* Fix a (commutative) ring R (with unity) and (left) R-modules M, N.

(a) Define: (i) an *R-module homomorphism* $\varphi : M \to N$; and (ii) the *kernel* $\ker \varphi$.

(b) Suppose $\varphi : M \to N$ is a homomorphism. Prove **one** of the following (both are true).

(i) $\ker \varphi$ is a submodule of M.

(ii) $\operatorname{Im} \varphi$ is a submodule of N.

(c) Prove that the *annihilator* of M, defined as

$$\operatorname{ann}(M) = \{r \in R : rm = 0 \text{ for all } m \in M\},$$

is an ideal of R.

(d) Prove that $\operatorname{ann}(R/I) = I$ for any ideal I.

(D2) *Generators.* Fix a (commutative) ring R (with unity) and a (left) R-module M.

(a) Given ring elements $a_1, \ldots, a_k \in R$, recall the definition of

$$\langle a_1, \ldots, a_k \rangle = \{\underline{\hspace{10cm}}\} \subseteq R,$$

the *ideal generated by* $a_1, \ldots, a_k$.

(b) Given elements $m_1, \ldots, m_k \in M$, decide on a definition of

$$\langle m_1, \ldots, m_k \rangle = \{\underline{\hspace{10cm}}\} \subseteq M,$$

the *submodule of M generated by* $m_1, \ldots, m_k$. Your answer should be the **smallest** submodule of M containing $m_1, \ldots, m_k$.

(c) Find the smallest, simplest possible generating set of R as an R-module (your answer will look the same regardless of what ring R is).

(d) Find the smallest, simplest possible generating set of $R \oplus R \oplus R$ as an R-module (one might be tempted to call this the *standard* generating set of $R \oplus R \oplus R$).

(D3) *Generators and relations.* Let $R = \mathbb{Q}[x, y]$, and let $e_1 = (1, 0)$, $e_2 = (0, 1) \in R \oplus R$. Let $\varphi : R \oplus R \to R$ denote the R-module homomorphism with $e_1 \mapsto x^3$ and $e_2 \mapsto y^2$.

(a) Find $\varphi(1, 2)$, $\varphi(xy, y^2)$, and $\varphi(x^2 + 2y, y^5 + 2y + 7)$.

(b) Find generators for the kernel and image of φ. Justify your claims.

(c) In what follows, let $M = R/\operatorname{Im} \varphi$. Determine $\dim_{\mathbb{Q}}(M)$, and find a $\mathbb{Q}$-basis for M.

(d) Let $I = \langle x, y \rangle \subset R$. Determine which elements $m \in M$ satisfy $I \cdot m = 0$.

(e) Locate an R-module homomorphism $\psi : R \oplus R \to R$ such that (i) $e_1, e_2 \notin \ker \psi$, and (ii) $R/\operatorname{Im} \psi$ is not a finite dimensional vector space over $\mathbb{Q}$.

(f) Determine whether $(R \oplus R)/\ker \varphi \cong R$ as R-modules.

(D4) *Quotient modules.* Fix a (commutative) ring R (with unity) and a (left) R-module M.

(a) Prove that if $R = \mathbb{Z}$ and $5 \in \operatorname{ann}(M)$, then M is ("naturally") a $\mathbb{Z}_5$-module.

(b) Given an ideal $I \subset R$, formulate a condition under which M is an R/I module.

(c) Given an ideal $I \subset R$, prove IM is a submodule of M.

(d) Determine $\operatorname{ann}(M/IM)$. What can we conclude when combined with part (b)?

(e) Find a $\mathbb{Z}_6$-module with 4 elements. Hint: first find one with 2 elements.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Fix rings R and T, a ring homomorphism $\varphi : R \to T$, and a T-module M. Prove that M is ("naturally") an R-module via the action $r \cdot m = \varphi(r)m$.

(H2) Fix a ring R and an R-module M, and fix $m \in M$. Prove that there exists a unique R-module homomorphism $\varphi : R \to M$ satisfying $\varphi(1) = m$.

(H3) Let $I = \langle x, y \rangle \subset R = \mathbb{Q}[x, y]$, and fix an R-module M and elements $m, m' \in M$.

- (a) Determine the precise condition on m and m' under which there exists an R-module homomorphism $\varphi : I \to M$ satisfying $\varphi(x) = m$ and $\varphi(y) = m'$.
- (b) Prove that when such a homomorphism φ exists, it is unique.

(H4) Determine whether each of the following statements is true or false. Prove your assertions.

- (a) Fix a ring R. Any R-module homomorphism $R \oplus R \to R$ must have nontrivial kernel.
- (b) Given any $\mathbb{Z}$-module M, there exists a unique way to extend the $\mathbb{Z}$-action on M to a $\mathbb{Q}$-action that makes M into a $\mathbb{Q}$-module.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Let $R = \mathbb{Q}[x, y]$, and let $I = \langle x^3, xy, y^2 \rangle \subset R$. Locate free modules F_0, F_1, and F_2 along with homomorphisms

$$0 \longrightarrow F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} R/I \longrightarrow 0$$

such that φ_0 is surjective, φ_2 is injective, $\ker \varphi_0 = \operatorname{Im} \varphi_1$, and $\ker \varphi_1 = \operatorname{Im} \varphi_2$.