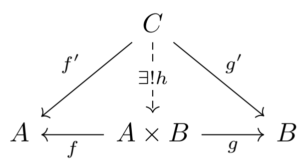
Fall 2021, Math 620: Week 15 Problem Set
Due: Friday, December 10th, 2021
Categories and Universal Properties

Discussion problems. The problems below should be worked on in class.

(D1) *Products.* Recall the *product* of two objects A and B in a category $\mathcal{C}$ is an object $A \times B$ together with morphisms $f : A \times B \to A$ and $g : A \times B \to B$ such that the following "universal property" is satisfied: for any object C in $\mathcal{C}$ with morphisms $f' : C \to A$ and $g' : C \to B$, there exists a unique morphism $h : C \to A \times B$ with $f' = f \circ h$ and $g' = g \circ h$ (we say the morphisms f' and g' "factor through" $A \times B$).



(a) In the category Ab, prove that (categorical) products are simply Cartesian products.

(b) Adjust your proof to argue that products in Set are Cartesian products.

(c) Fix an index set $\mathcal{I}$ and a collection of objects A_i for $i \in \mathcal{I}$ in a category $\mathcal{C}$. Give a definition of the *product* $\prod_{i\in\mathcal{I}} A_i$. Use both words and a commutative diagram (you may find it useful to have a separate diagram for each element of $\mathcal{I}$).

(D2) *Kernels.* The *kernel* of a morphism $f : A \to B$ in a category $\mathcal{C}$ is an object K together with a morphism $g : K \to A$ satisfying $f \circ g = 0$, such that the following universal property holds: for any object K' and morphism $g' : K' \to A$ with $f \circ g' = 0$, there exists a unique morphism $h : K' \to K$ such that $g \circ h = g'$.

(a) Draw the commutative diagram for kernels.

(b) Let $\varphi : \mathbb{Z} \to \mathbb{Z}_{12}$ denote the group homomorphism given by $\varphi(a) = [4a]_{12}$. Prove that $K = \ker \varphi$ with the inclusion map $g : K \hookrightarrow \mathbb{Z}$ satsifies the universal property above.

(c) Prove that every morphism in the category Ab has a (categorical) kernel that coincides with what we have been calling the "kernel" all semester.

(D3) *Duals.* Given a universal object (e.g., kernels, products), the *dual* is obtained by reversing the directions of all of the arrows in the accompanying commutative diagram.

(a) Write the definition of *coproduct* (the categorical dual of the product) of two objects A and B in a category $\mathcal{C}$. Denote this object by $A \amalg B$.

(b) In the category Ab, determine what familiar group $\mathbb{Z} \amalg \mathbb{Z}_6$ is.

(c) Demonstrate that in the category Ab, finite coproducts are simply finite products.

(d) Define

$$G = \{(a_1, a_2, \ldots) : a_i \in \mathbb{Z} \text{ and } a_i = 0 \text{ for all but finitely many } i\}.$$

Argue that G is a subgroup of $\prod_{i=1}^{\infty} \mathbb{Z} = \{(a_1, a_2, \ldots) : a_i \in \mathbb{Z}\}$.

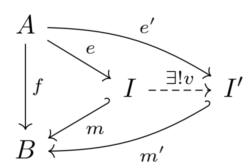
(e) Demonstrate that $\prod_{i=1}^{\infty} \mathbb{Z}$ does **not** satisfy the universal property for $\coprod_{i=1}^{\infty} \mathbb{Z}$ in Ab. Hint: use the group G from the previous part.

(f) Formulate and prove a conjecture for what objects are coproducts in Ab.
Hint: where have we seen the group G before?

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Given a category $\mathcal{C}$ and a morphism $f : A \to B$, the *image* of f (if it exists) is an object I together with a monomorphism $m : I \to B$ such that

- there exists a map $e : A \to I$ such that $m \circ e = f$; and
- the following universal property is satisfied: for any object I', morphism $e' : A \to I'$, and monomorphism $m' : I' \to B$ satisfying the above requirements, there exists a unique morphism $v : I \to I'$ such that the following diagram commutes.



Prove that for any ring R in Slt, the category R-Mod has images.

(H2) Prove that in any category $\mathcal{C}$ in which every morphism has a kernel (as defined in Problem (D1) from discussion), the morphism g must be a monomorphism.

Be careful: morphisms might not be "functions" in general!

(H3) Given a morphism $f : A \to B$ in a category $\mathcal{C}$, its *cokernel* satisfies (as the name suggests) the dual universal property to that of the kernel.

(a) Write the full definition of a *cokernel* of a morphism $f : A \to B$ in a category $\mathcal{C}$.

Be sure to include a diagram!

(b) Detemine whether the category Ab has cokernels.

(H4) Determine whether each of the following statements is true or false. Prove your assertions.

(a) The category Slt has kernels.

(b) The category Set has coproducts.