Fall 2022, Math 522: Week 8 Problem Set
Due: Friday, October 28th, 2022
The Möbius Function

Discussion problems. The problems below should be worked on in class.

(D1) *Möbius inversion formula.* We will be using, without proof for the moment, the following.

Theorem (Möbius inversion formula)**.** *Two functions* $f, g : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ *satisfy*

$$f(n) = \sum_{d|n} g(d)$$

for every $n \in \mathbb{Z}_{\geq 1}$ *if and only if, for every* $n \in \mathbb{Z}_{\geq 1}$,

$$g(n) = \sum_{d|n} \mu(d) f(n/d).$$

(a) For each $n = 1, \ldots, 10$, find

$$\sum_{d|n} \mu(d) d\left(\frac{n}{d}\right).$$

Formulate a conjecture for general n. (No proof needed yet!)

(b) Use the Möbius inversion formula to prove your conjecture.
Hint: pick the functions f and g so that the conclusion of the Möbius inversion formula matches the equality you wish to prove.

(c) Use the Möbius inversion formula to find a formula (in terms of n) for

$$\sum_{d|n} \mu(d) \sigma\left(\frac{n}{d}\right).$$

Hint: begin by finding this sum for $n = 1, \ldots, 10$.

(d) In class last week, we proved the first equality in

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right) = \sum_{d|n} \mu(n) \frac{n}{d}.$$

Use the Möbius inversion formula to prove the second equality.

(D2) *Proving the Möbius inversion formula.* The majority of the proof comes from carefully examining the following algebraic manipulation, which we will justify in several steps.

$$\begin{aligned}\sum_{d|n} \mu(d) f(n/d) &= \sum_{\substack{d,d' \leq n \\ d \cdot d' = n}} \mu(d) f(d') = \sum_{\substack{d,d' \leq n \\ d \cdot d' = n}} \mu(d) \sum_{e|d'} g(e) \\ &= \sum_{\substack{d,e,h \leq n \\ d \cdot e \cdot h = n}} \mu(d) g(e) = \sum_{\substack{e,h' \leq n \\ e \cdot h' = n}} g(e) \sum_{d|h'} \mu(d).\end{aligned}$$

(a) For $n = 12$, write out each step **without** sigma-sums. Verify each equality in this case.

(b) Give a thorough written justification of each step of the above algebra.

(c) Explain why the final expression above equals $g(n)$, thereby completing the proof of the forward direction.

(d) Using a similar argument, prove the converse direction of the Möbius inversion formula (that is, if the second equality holds for all n, then so does the first).

Homework problems. You must submit *all* homework problems in order to receive full credit.

Unless otherwise stated, $a, b, c, n, p \in \mathbb{Z}$ are arbitrary with $p > 1$ prime and $n \geq 2$.

(H1) Draw the divisibility poset of $n = 441$. Try your best to obtain the most visually appealing drawing you can (this may take several attempts). Write the value $\mu(d)$ next to each element d.

Hint: $441 = 3^2 7^2$.

(H2) Prove that there are infintely many integers n such that $\phi(n)$ is a perfect square.

(H3) (a) Prove that if $a \mid b$ then $\phi(a) \mid \phi(b)$.

(b) Given $n \geq 2$, conjecture a formula for

$$\sum_{d \mid n} (\mu(d))^2 \frac{\phi(n)}{\phi(d)}$$

in terms of n. Demonstrate your conjecture holds for at least 5 consecutive values of n. How is part (a) relevant to this problem?

Note: you are **not** required to prove your conjecture.

(H4) Prove that for all $n \geq 2$,

$$\sum_{d \mid n} \mu(d)\phi(d) = \prod_{p \mid n} (2 - p).$$

Hint: induct on the number of distinct primes in the factorization of n.

(H5) Determine whether each of the following is true or false. Prove each true statement, and give a counterexample for each false statement.

(a) If $a \mid b$ and $\phi(a) = \phi(b)$, then $a = b$.

(b) For all $n \geq 2$, the quantity

$$\sum_{d \mid n} \frac{\phi(d) + 1}{d}$$

is an integer.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Prove your formula from Problem (H3)(b).

(C2) Let D_n denote the set of positive divisors of n. A function $f : D_n \to \mathbb{Z}$ is called *order preserving* if $f(a) \leq f(b)$ whenever $a, b \in D_n$ satisfy $a \mid b$. For each $t \geq 1$, let $L(t)$ denote the number of order preserving functions $D_n \to \{1, \ldots, t\}$.

Prove that for any fixed n, the funtion $L(t)$ is a polynomial in t of degree $d(n)$.