Fall 2022, Math 522: Week 10 Problem Set
Due: Friday, November 18th, 2022
Locating Large Primes

Discussion problems. The problems below should be worked on in class.

(D1) *Treading water in hard-problem infested seas.* The following are claims someone might walk up to you on the street and make. Use intuition about prime numbers, as discussed in class on Monday/Wednesday, to determine whether the statements are likely true or likely false. Give a detailed, intuitive justification of your position.

After completing the first 3 parts, and after completing all parts, verify with James that you settled on the correct truth value for each.

(a) For every $n \geq 2$, each equivalence class modulo n contains infinitely many primes.

(b) For every $n \geq 2$, each equivalence class modulo n that contains at least one prime contains infinitely many primes.

(c) For every $n \geq 2$, each equivalence class modulo n that contains at least two primes contains infinitely many primes.

(d) For every prime $p > 1000$, the number $p^p + 2$ is also prime.

(e) For every $k \geq 1$, the interval $[100k, 100k + 99]$ (i.e., each range 100-199, 200-299, etc.) contains at least one prime.

(f) There exist arbitrarily large gaps between sequential primes.

(g) There exist infinitely many integers n such that n, $n + 2$, and $n + 4$ are all prime (we might want to call this the "triplet prime conjecture").

(D2) *And now, some proofs.* With the exception of part (d), every statement in Problem (D1) **can** be proven (or disproven) with the tools obtained on Monday/Wednesday. Do so now.

Be sure you have verified with James that you settled on the correct truth value for each before attempting a proof!

(D3) *A weaker version of the Prime Number Theorem.* The Prime Number Theorem states

$$\lim_{n\to\infty} \frac{\pi(n)}{n/\log(n)} = 1.$$

In this problem, we will prove a weaker result, namely that

$$\pi(n) \geq \log_2(\log_2(n))$$

for every $n \geq 2$.

(a) For each $n \geq 1$, let $a_n = 2^{2^n} + 1$. Prove that if $n < m$, then $a_n \mid (a_m - 2)$

(b) Use part (a) to prove that if $n \neq m$, then $\gcd(a_n, a_m) = 1$.

(c) Conclude that there are at least n primes p such that $p \leq a_n$.

(d) Conclude that $\pi(n) \geq \log_2(\log_2(n))$.

Homework problems. You must submit *all* homework problems in order to receive full credit.

Unless otherwise stated, $a, b, c, n, p \in \mathbb{Z}$ are arbitrary with $p > 1$ prime and $n \geq 2$.

(H1) Do all parts of Problem (D3) from the discussion.

(H2) For this problem, you may **not** use Dirichlet's theorem.

(a) Prove that infinitely many primes p satisfy $p \equiv 3 \bmod 4$.
Hint: assume $p_1, \ldots, p_k$ are all such primes, and consider $n = 4p_1 \cdots p_k - 1$.

(b) Prove that infinitely many primes p satisfy $p \equiv 1 \bmod 4$.
Hint: assume $p_1, \ldots, p_k$ are all such primes, and consider $n = 4p_1^2 \cdots p_k^2 + 1$.
You may find the following fact helpful (from Problem (H5)(b) on Homework Set 9): given $n \geq 1$, any odd prime p dividing $n^2 + 1$ must satisfy $p \equiv 1 \bmod 4$.

(H3) Determine whether each of the following is true or false. Prove each true statement, and give a counterexample for each false statement.

(a) For each $k \geq 1$, there exists an n such that $n, n+1, \ldots, n+k$ are all composite.

(b) For every $n \geq 1$, the integer $n^2 - n + 41$ is prime.

(c) If n is composite, then $2^n - 1$ is also composite.

(d) If n is prime, then $2^n - 1$ is also prime.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Read the proof of Tchebychev's Theorem in Andrews 8.2, and write a "roadmap" for the proof (i.e., a summary of the main steps, without including any algebraic details).