Fall 2022, Math 522: Week 14 Problem Set
Polynomial Arithmetic

Discussion problems. The problems below should be worked on in class.

(D1) *Polynomial long division.*

(a) Divide $a(x) = x^4 + x^3 + 2x^2 + x + 1$ by $b(x) = x^2 + 1$ over $\mathbb{Q}$.

(b) Divide $a(x) = 2x^5 - x^4 + 3x^3 + 2x^2 + x + 1$ by $b(x) = 2x^2 + x + 1$ over $\mathbb{Q}$.

(c) Find $\gcd(84, 32)$ using the Euclidean algorithm. Note: this is a week 2 question!

(d) Use the Euclidean algorithm to find the greatest common divisor (over $\mathbb{Q}$) of

$$f(x) = x^4 - x^3 - 2x - 4 \qquad \text{and} \qquad g(x) = x^4 - 2x^3 + 3x^2 - 4x + 2.$$

Remember: the greatest common divisor of two polynomials must be monic!

(D2) *Similarities between $F[x]$ and $\mathbb{Z}$.* In what follows, assume $F = \mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$.

(a) Below is a (correct!) proof that if $a, b, c \in \mathbb{Z}$ with $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$. Copy it onto the board. Then, prove that if $a(x), b(x), c(x) \in F[x]$ with $a(x) \mid b(x)c(x)$ and $\gcd(a(x), b(x)) = 1$, then $a(x) \mid c(x)$.

Proof. Since $a \mid bc$ and $\gcd(a, b) = 1$, there exist $m \in \mathbb{Z}$ and $x, y \in \mathbb{Z}$ satsifying $am = bc$ and $ax + by = 1$. As such,

$$c = acx + bcy = acx + amy = a(cx + my),$$

so $a \mid c$. $\square$

(b) Fill in the gaps in the proof that if $a, b, c \in \mathbb{Z}$ with $c > 0$, then $\gcd(ca, cb) = c\gcd(a, b)$. Identify where the hypothesis $c > 0$ is used.

Proof. Let $d = \gcd(a, b)$, so $a = md$ and $b = nd$ for some $m, n \in \mathbb{Z}$. This means _______ and _______, so $cd \mid ca$ and $cd \mid cb$. Moreover, $ax + by = d$ for some $x, y \in \mathbb{Z}$, so _______, meaning $cd = \gcd(ca, cb)$. $\square$

(c) State and prove an analogous result to part (b) for elements of $F[x]$.
Remember: the greatest common divisor of two polynomials must be monic!

(d) Complete the following proof that if $a(x), b(x) \in F[x]$ satisfy $a(x) \mid b(x)$ and $b(x) \mid a(x)$, then $b(x) = Ca(x)$ for some $C \in F$.

Proof. Since $a(x) \mid b(x)$, we have $b(x) = a(x)f(x)$ for some $f(x) \in F[x]$, and since $b(x) \mid a(x)$, we have ______________. This means

$$\deg b(x) = \deg f(x) + \deg a(x) \geq \deg a(x) = \underline{\hspace{3cm}} \geq \deg b(x),$$

so $\deg b(x) = \deg$ ____ and $\deg f(x) = 0$. Choosing $C =$ ____ completes the proof. $\square$