Fall 2023, Math 320: Week 0 Problem Set
Due: Thursday, August 31st, 2023
Proof Writing Review

Discussion problems. The problems below should be worked on in class.

(D1) (a) Consider the following statement.

"If $a+b$ and ab are both even, then a and b are both even."

Locate the error in the following proof of the statement. Then write a correct proof.

Proof. We prove the contrapositive of the statement. Suppose it is **not** the case that a and b are both even. This means a and b are both odd. Necessarily, $a+b$ is even, but ab is odd. As such, $a+b$ and ab are not both even. This completes the proof. □

(b) Determine whether each of the following statements is true or false. Prove your claims.

(i) The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 9x^2 - 6x + 1$ is injective (one-to-one).

(ii) The function $f : \mathbb{Z} \to \mathbb{R}$ given by $f(x) = 9x^2 - 6x + 1$ is injective (one-to-one).

(c) Let $D = \{(a,b) : a,b \in \mathbb{Z}\}$, and define a relation $\sim$ on D with $(a,b) \sim (a',b')$ whenever $a \le a'$ and $b \le b'$. Is $\sim$ reflexive? Is $\sim$ symmetric? Is $\sim$ transitive? Prove your claims.

(D2) (a) Fill in the blanks in the following proof that for every $n \ge 1$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Proof. We proceed by induction on n.
Base case: $n = 1$. We obtain $1^2 =$ ____ $= \frac{1(2)(3)}{6}$, as expected.
Inductive step: assuming $n \ge 1$ and

$$\underline{\hspace{10em}} = \frac{n(n+1)(2n+1)}{6},$$

we wish to show

$$1^2 + 2^2 + \cdots + n^2 + (n+1)^2 = \underline{\hspace{10em}}.$$

To this end, we see that

$$\begin{aligned} 1^2 + 2^2 + \cdots + n^2 + (n+1)^2 &= \underline{\hspace{10em}} \\ &= \underline{\hspace{10em}} \end{aligned}$$

as desired. □

(b) Suppose $f : \mathbb{Z}_{\ge 0} \to \mathbb{Z}$ satisfies $f(0) = 1$, $f(1) = 2$, and

$$f(n) = 2f(n-1) - f(n-2)$$

for $n \ge 2$. Find a formula for $f(n)$ by experimentation, then prove it using induction. Write your proof **without** introducing any new variables (i.e., just using n), and be **very** careful when stating your inductive hypothesis!

(c) Use induction on n to prove that

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$$

for all $n \ge 1$.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Suppose $x, y \in \mathbb{R}$. Prove $x^3 + x^2y = y^2 + xy$ if and only if $y = x^2$ or $y = -x$.

(H2) Suppose $x \in \mathbb{R}$ with $x > 0$.

- (a) Use induction on n to prove that

$$(1 + x)^n \geq 1 + nx$$

  for all $n \in \mathbb{Z}$ with $n \geq 1$.

  Note: you may **not** use the binomial theorem in your proof.

- (b) Is part (a) still true if the "$x > 0$" assumption is omitted? If not, where in your proof did you use this assumption?

(H3) Let $D = \{(a, b) : a, b \in \mathbb{Z}\}$, and define a relation $\sim$ on D with $(a, b) \sim (a', b')$ whenever $a + b = a' + b'$.

- (a) Prove $\sim$ is an equivalence relation (that is, $\sim$ is reflexive, symmetric, and transitive).
- (b) Find the equivalence class of $(0, 0)$.

(H4) Determine whether each of the following statements is true or false. Prove your claims.

- (a) If a polynomial $f : \mathbb{R} \to \mathbb{R}$ is injective, then f is surjective.
- (b) If a polynomial $f : \mathbb{Z} \to \mathbb{Z}$ is injective, then f is surjective.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Let $D = \{(a, b) : a, b \in \mathbb{Z}_{\geq 0}\}$. Prove that the function $f : D \to \mathbb{Z}_{\geq 0}$ given by

$$f(a, b) = \tfrac{1}{2}(a + b)(a + b + 1) + a$$

is a bijection (that is, f is one-to-one and onto).

Hint: as you begin working on this (i.e., in your scratchwork), you may find it helpful to start by drawing a picture of the set D (as points in the plane) and write $f(a, b)$ next to each point (a, b) for $a, b \leq 5$.