Fall 2023, Math 320: Week 4 Problem Set
Due: Thursday, September 28th, 2023
Introduction To Rings

Discussion problems. The problems below should be worked on in class.

(D1) *Checking ring axioms.* Determine which of the following sets R is a ring under the given addition and multiplication (hint: in some parts, Theorem 3.2 may be useful). For each ring, determine whether it is (i) commutative, (ii) an integral domain, and (iii) a field.

(a) The set R of 2×2 real matrices (under matrix addition/multiplication) given by

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{R} \right\} \subset M_2(\mathbb{R}).$$

First, fill in the blanks in the following proof that R is closed under multiplication.

Proof. For $M, N \in R$, where $M = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, $N = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix}$ for some $a, b, c, a', b', c' \in \mathbb{R}$,

$$MN = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} = \begin{pmatrix} \underline{\qquad} & \underline{\qquad} \\ 0 & \underline{\qquad} \end{pmatrix} \in R,$$

so R is closed under multiplication. $\square$

(b) The set $R = \{r_5x^5 + \cdots + r_1x + r_0 : r_i \in \mathbb{R}\} \subset \mathbb{R}[x]$ of polynomials in a variable x with real coefficients and **degree at most 5**, under the usual addition and multiplication.

(c) The set $R = \mathbb{R} \cup \{\infty\}$ of real numbers together with infinity, and addition and multiplication operations $a \oplus b = \min(a, b)$ and $a \odot b = a + b$, respectively.

(d) The set $R = \mathbb{Z}$ with operations $\oplus$ and $\odot$ given by $a \oplus b = a + b$ and $a \odot b = a + b$ (in particular, **both** addition and multiplication in R correspond to integer addition).

(e) The set $R = \{p(x) \in \mathbb{R}[x] : p(0) \in \mathbb{Z}\}$ of polynomials in a variable x with real coefficients and **integer constant term**, under the usual addition and multiplication. For example, $2x^2 + \frac{1}{2}x + 5 \in R$ and $\frac{6}{5}x \in R$, but $5x + \frac{1}{3} \notin R$.

(f) The set $R = \mathbb{R}_{>0}$ of positive real numbers with operations $\oplus$ and $\odot$ given by $a \oplus b = ab$ and $a \odot b = a^{\ln(b)}$ for all $a, b \in R$.

(D2) *Cartesian products.* The Cartesian product of two rings R_1 and R_2 is the set

$$R_1 \times R_2 = \{(a, b) : a \in R_1, b \in R_2\}$$

with addition $(a, b) + (a', b') = (a + a', b + b')$ and multiplication $(a, b) \cdot (a', b') = (a \cdot a', b \cdot b')$. Note: the operation in each coordinate happen in their respective rings.

(a) Find $(1, 2) + (3, 4)$ and $(1, 2) \cdot (3, 4)$ in $\mathbb{R} \times \mathbb{R}$. Locate the additive identity of $\mathbb{R} \times \mathbb{R}$.

(b) Find the additive inverses of $([1]_6, [0]_3)$, $([3]_6, [2]_3)$, and $([5]_6, [1]_3) \in \mathbb{Z}_6 \times \mathbb{Z}_3$.

(c) What is the multiplicative identity of $\mathbb{Z}_6 \times \mathbb{Z}_3$? Which elements listed in part (b) have a multiplicative inverse?

(d) Justify each "=" in the following proof that addition is commutative in $R_1 \times R_2$ for any rings R_1 and R_2.

Proof. Given $(a, b), (c, d) \in R_1 \times R_2$, we have

$$(a, b) + (c, d) = (a + c, b + d) = (c + a, b + d) = (c + a, d + b) = (c, d) + (a, b),$$

which completes the proof. $\square$

(e) Prove that every element of $R_1 \times R_2$ has an additive inverse.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Locate a subring $R \subseteq \mathbb{Z}_{24}$ with exactly 3 elements (be sure to prove that your chosen subset R is indeed a subring of $\mathbb{Z}_{24}$). Does R have a multiplicative identity?

(H2) (a) Locate a subring of $\mathbb{Z}_5 \times \mathbb{Z}_4$ with exactly 5 elements.

(b) Locate a subring of $\mathbb{Z}_6 \times \mathbb{Z}_4$ with exactly 12 elements.

(H3) Let

$$R = \left\{ B \in M_2(\mathbb{R}) : \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

(a) Locate 3 distinct matrices in R.

(b) Prove that R is a subring of $M_2(\mathbb{R})$.

(H4) Let $R = \mathbb{Z}$ and define

$$a \oplus b = a + b + 1 \qquad \text{and} \qquad a \odot b = ab + a + b$$

for all $a, b \in R$. Prove that $(R, \oplus, \odot)$ is a commutative ring.

(H5) Determine whether each of the following statements is true or false. Prove each true statement, and give a counterexample for each false statement.

(a) If R is a ring and S is a subset of R, then S is a subring of R.

(b) The ring $\mathbb{Z} \times \mathbb{Z}$ is an integral domain.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Let S be a set and $P(S)$ denote the set of all subsets of S. Define addition and multiplication operations $\oplus$ and $\odot$ by

$$M \oplus N = (M \setminus N) \cup (N \setminus M) \qquad \text{and} \qquad M \odot N = M \cap N$$

for all $M, N \in P(S)$. Determine whether $(P(S), \oplus, \odot)$ is a field.