Fall 2023, Math 320: Week 5 Problem Set
Due: Thursday, October 5th, 2023
Properties of Rings

Discussion problems. The problems below should be worked on in class.

(D1) *The ring structure of* $\mathbb{Z}_n$. The goal of this problem is to determine which elements of $\mathbb{Z}_n$ are zero-divisors, which are units, and which are neither.

(a) Compare your answers to problem (P1). Then find all zero divisors in $\mathbb{Z}_5$, $\mathbb{Z}_8$, and $\mathbb{Z}_{10}$.

(b) Prove that if $m, n \geq 2$, then $\mathbb{Z}_{mn}$ is not an integral domain.

(c) Suppose p is prime. Complete the following proof that $\mathbb{Z}_p$ is an integral domain.
Hint: use the fact that if p is prime and $p \mid ab$, then $p \mid a$ or $p \mid b$.

Proof. Fix $[a]_p, [b]_p \in \mathbb{Z}_p$, and suppose $[a]_p[b]_p = [0]_p$. We need to prove that either $[a]_p = [0]_p$ or $[b]_p = [0]_p$. Since $[0]_p = [a]_p[b]_p = [ab]_p$, we have $p \mid ab$. As such, ... □

(d) Multiply each element of $\mathbb{Z}_7$ by $[4]_7$ (i.e. find $[0]_7 \cdot [4]_7$, then $[1]_7 \cdot [4]_7$, and so forth). Do the same with $[5]_7$. What do you notice about which elements of $\mathbb{Z}_7$ appear?

(e) Multiply every element of $\mathbb{Z}_{11}$ by $[3]_{11}$. Which elements of $\mathbb{Z}_{11}$ are obtained?
Hint: you may want to "divide and conquer" within your group!

(f) Suppose p is prime. Find and correct the error in the following proof that $\mathbb{Z}_p$ is a field.

Proof. Fix an arbitrary $a \in \mathbb{Z}_p$. Since $\mathbb{Z}_p$ is finite, let $a_1, a_2, \ldots, a_p$ denote the complete list of distinct elements of $\mathbb{Z}_p$. We must find k so that $a_k \cdot a = [1]_p$. Consider the list

$$a_1 \cdot a, \quad a_2 \cdot a, \quad \ldots, \quad a_p \cdot a.$$

We claim these elements are all distinct from one another: since $\mathbb{Z}_p$ is an integral domain, if $a_i \cdot a = a_j \cdot a$, then cancelling the a's yields $a_i = a_j$. This means every element of $\mathbb{Z}_p$ appears exactly once in the centered list. In particular, $[1]_p$ appears somewhere in the list, meaning for some k, we have $a_k \cdot a = [1]_p$. □

(g) Look carefully at the proof in part (f). What properties of $\mathbb{Z}_p$ were used in the proof? Use this to complete the following (much more general) result.

Theorem (Theorem 3.9). *If R is a integral domain and __________, then R is a field.*

(h) Combining the results above, characterize (in terms of n) when $\mathbb{Z}_n$ is a field, when $\mathbb{Z}_n$ is an integral domain but not a field, and when $\mathbb{Z}_n$ is neither an integral domain nor a field. State your characterization formally (as a theorem), and **box it 3 times**.

(D2) *Cartesian products.*

(a) Determine which elements of $\mathbb{Z}_5 \times \mathbb{Z}_4$ are units, and which are zero-divisors.

(b) Suppose $m, n \geq 2$. Determine the units and zero-divisors of $\mathbb{Z}_m \times \mathbb{Z}_n$.

(c) Locate an element of $\mathbb{Z} \times \mathbb{Z}$ that is a zero-divisor, an element that is a unit, and a nonzero element that is neither a unit nor a zero-divisor.

(d) Suppose R_1 and R_2 are rings. Determine which elements of $R_1 \times R_2$ are units, in terms of the units of R_1 and the units of R_2.

(e) Suppose R_1 and R_2 are rings. Determine which elements of $R_1 \times R_2$ are zero-divisors, in terms of the zero-divisors of R_1 and the zero-divisors of R_2.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Consider

$$S = \left\{ \begin{pmatrix} a & a \\ b & b \end{pmatrix} \in M_2(\mathbb{R}) : a, b, \in \mathbb{R} \right\}$$

(a) Verify S is a subring of $M_2(\mathbb{R})$.

(b) Locate a matrix $J \in S$ that is a *right multiplicative identity* (that is, $AJ = A$ for every $A \in S$). Does S have a unity element?

(H2) Suppose R is a ring and $S, T \subseteq R$ are subrings. Prove $S \cap T$ is a subring of R.

(H3) Fix a **commutative** ring R. An element $r \in R$ is *nilpotent* if $r^n = 0$ for some $n \geq 1$.

(a) Prove that if $a, b \in R$ are nilpotent, then ab is nilpotent.

(b) Prove that if $a, b \in R$ are nilpotent, then $a + b$ is nilpotent.

(H4) Determine whether each of the following statements is true or false. Prove each true statement, and give a counterexample for each false statement.

(a) If R is a commutative ring and $a, b \in R$ are units, then ab is a unit.

(b) If R is a commutative ring and $a, b \in R$ are zero divisors, then ab is a zero divisor. Hint: this one is subtle!

(c) If R is a ring and $S, T \subseteq R$ are subrings, then $S \cup T$ is a subring of R.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Determine for which $m \geq 2$ the set of non-unit elements of $\mathbb{Z}_m$ is closed under both addition and multiplication. Prove your claim.