Fall 2023, Math 320: Week 6 Problem Set
Due: Thursday, October 12th, 2023
Arithmetic in Rings

Discussion problems. The problems below should be worked on in class.

(D1) *Ring arithmetic.* Suppose $(R, +, \cdot)$ is a ring. Try to use only one axiom (or theorem) in each proof step in this problem.

(a) Fill in the justifications in the proof that for every $a, b \in R$, $-(a + b) = (-a) + (-b)$.

Proof. By definition, $-(a+b)$ is the additive inverse of $a+b$. To prove that $(-a)+(-b)$ is the same element of R, we must prove that adding it to $a + b$ yields 0. Indeed,

$$\begin{aligned}(a + b) + ((-a) + (-b)) &= (a + b) + ((-b) + (-a)) && (\underline{\qquad\qquad\qquad}) \\ &= (a + (b + (-b))) + (-a) && (\underline{\qquad\qquad\qquad}) \\ &= (a + 0_R) + (-a) && (\underline{\qquad\qquad\qquad}) \\ &= a + (-a) && (\underline{\qquad\qquad\qquad}) \\ &= 0_R && (\underline{\qquad\qquad\qquad})\end{aligned}$$

and a similar argument shows $((-a) + (-b)) + (a + b) = 0_R$. □

(b) Prove that for every $a \in R$, $-(-a) = a$.

(c) Prove that for every $a, b \in R$, $-(a - b) = (-a) + b$.
Hint: $a - b$ means $a + (-b)$.

(d) Prove that if R has a unity, then for every $a \in R$, $(-1_R)a = -a$.

(e) Prove that if $a, b \in R$ with a and ab both units, then b is a unit.

(D2) *Identifying familiar rings in disguise.* Throughout this problem, let

$$R_1 = \mathbb{C} \qquad \text{and} \qquad R_2 = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

(a) Find the sum and product of $3 + 4i$, $5 + 6i \in R_1$.

(b) Find the sum and product of $\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}, \begin{pmatrix} 5 & 6 \\ -6 & 5 \end{pmatrix} \in R_2$.

(c) Compare your answers to parts (a) and (b). What do you notice?

(d) Find the sum and product of two arbitrary elements $a + bi, a' + b'i \in R_1$.

(e) Find the sum and product of two arbitrary elements $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \begin{pmatrix} a' & b' \\ -b' & a' \end{pmatrix} \in R_2$.

(f) Is there a natural way to "match" each element of R_1 with an element of R_2? Define a function $\varphi : R_1 \to R_2$ for this "matching".

(g) Using your observations above, find an equation relating $\varphi(c_1)$, $\varphi(c_2)$, and $\varphi(c_1 + c_2)$ for $c_1, c_2 \in R_1$. Hint: start with $c_1 = 3 + 4i$ and $c_2 = 5 + 6i$ as a guide.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Suppose $R = \{0_R, 1_R, a\}$ is a ring, and that a is a unit. Use the ring axioms to fill in the addition table and multiplication table of R (each entry should be either 0_R, 1_R, or a). Give a justification for each entry.

$+$	0_R	1_R	a
0_R			
1_R			
a			

$\cdot$	0_R	1_R	a
0_R			
1_R			
a			

(H2) Suppose $(R, +, \cdot)$ is a ring. Prove each of the following statements. Identify each ring axiom you use, and try to only use one axiom (or theorem) in each step.

(a) For any $a, b, c, d \in R$, we have $a - b + c = d$ if and only if $a = b - c + d$.

(b) If $a, b, c \in R$ with $ab = 1_R$ and $ca = 1_R$, then $b = c$.

(c) If R has unity and $1_R = 0_R$, then $R = \{0_R\}$.

(H3) Fix a ring R and a unit $a \in R$. Prove **by induction** $(a^{-1})^n = (a^n)^{-1}$ for every $n \in \mathbb{Z}_{\geq 1}$.

(H4) Determine whether each of the following statements is true or false. Prove each true statement, and give a counterexample for each false statement.

(a) If R is a ring, then every element of R is either a unit or a zero-divisor.

(b) If F_1 and F_2 are fields, then $F_1 \times F_2$ is a field.

(c) If $a, b \in R$ are units, then $a + b$ is a unit.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Suppose $R = \{0_R, 1_R, a, b\}$ is a field. Use the field axioms to fill in the addition table and multiplication table of R (each entry should be either 0_R, 1_R, a, or b). Give a justification for each entry (except the one that has been filled in for you).

$+$	0_R	1_R	a	b
0_R				
1_R				
a		b		
b				

$\cdot$	0_R	1_R	a	b
0_R				
1_R				
a				
b				