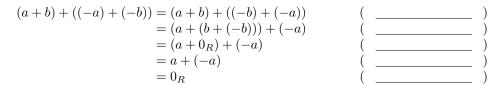
Fall 2023, Math 320: Week 6 Problem Set Due: Thursday, October 12th, 2023 Arithmetic in Rings

Discussion problems. The problems below should be worked on in class.

- (D1) Ring arithmetic. Suppose $(R, +, \cdot)$ is a ring. Try to use only one axiom (or theorem) in each proof step in this problem.
 - (a) Fill in the justifications in the proof that for every $a, b \in R$, -(a + b) = (-a) + (-b).

Proof. By definition, -(a+b) is the additive inverse of a+b. To prove that (-a)+(-b) is the same element of R, we must prove that adding it to a+b yields 0. Indeed,



and a similar argument shows $((-a) + (-b)) + (a + b) = 0_R$.

- (b) Prove that for every $a \in R$, -(-a) = a.
- (c) Prove that for every $a, b \in R$, -(a b) = (-a) + b. Hint: a - b means a + (-b).
- (d) Prove that if R has a unity, then for every $a \in R$, $(-1_R)a = -a$.
- (e) Prove that if $a, b \in R$ with a and ab both units, then b is a unit.
- (D2) Identifying familiar rings in disguise. Throughout this problem, let

$$R_1 = \mathbb{C}$$
 and $R_2 = \left\{ \left(\begin{array}{cc} a & b \\ -b & a \end{array} \right) : a, b \in \mathbb{R} \right\}.$

(a) Find the sum and product of 3 + 4i, $5 + 6i \in R_1$.

(b) Find the sum and product of
$$\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$$
, $\begin{pmatrix} 5 & 6 \\ -6 & 5 \end{pmatrix} \in R_2$.

- (c) Compare your answers to parts (a) and (b). What do you notice?
- (d) Find the sum and product of two arbitrary elements $a + bi, a' + b'i \in R_1$.
- (e) Find the sum and product of two arbitrary elements $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, $\begin{pmatrix} a' & b' \\ -b' & a' \end{pmatrix} \in R_2$.
- (f) Is there a natural way to "match" each element of R_1 with an element of R_2 ? Define a function $\varphi: R_1 \to R_2$ for this "matching".
- (g) Using your observations above, find an equation relating $\varphi(c_1)$, $\varphi(c_2)$, and $\varphi(c_1 + c_2)$ for $c_1, c_2 \in R_1$. Hint: start with $c_1 = 3 + 4i$ and $c_2 = 5 + 6i$ as a guide.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Suppose $R = \{0_R, 1_R, a\}$ is a ring, and that a is a unit. Use the ring axioms to fill in the addition table and multiplication table of R (each entry should be either 0_R , 1_R , or a). Give a justification for each entry.

+	0_R	1_R	a		•	0_R	1_R	a
0_R				_	0_R			
1_R				_	1_R			
a				_	a			

- (H2) Suppose $(R, +, \cdot)$ is a ring. Prove each of the following statements. Identify each ring axiom you use, and try to only use one axiom (or theorem) in each step.
 - (a) For any $a, b, c, d \in R$, we have a b + c = d if and only if a = b c + d.
 - (b) If $a, b, c \in R$ with $ab = 1_R$ and $ca = 1_R$, then b = c.
 - (c) If R has unity and $1_R = 0_R$, then $R = \{0_R\}$.
- (H3) Fix a ring R and a unit $a \in R$. Prove by induction $(a^{-1})^n = (a^n)^{-1}$ for every $n \in \mathbb{Z}_{>1}$.
- (H4) Determine whether each of the following statements is true or false. Prove each true statement, and give a counterexample for each false statement.
 - (a) If R is a ring, then every element of R is either a unit or a zero-divisor.
 - (b) If F_1 and F_2 are fields, then $F_1 \times F_2$ is a field.
 - (c) If $a, b \in R$ are units, then a + b is a unit.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Suppose $R = \{0_R, 1_R, a, b\}$ is a field. Use the field axioms to fill in the addition table and multiplication table of R (each entry should be either $0_R, 1_R, a, \text{ or } b$). Give a justification for each entry (except the one that has been filled in for you).

+	0_R	1_R	a	b	•	0_R	1_R	a	b
0_R					0_R				
1_R					1_R				
a		b			a				
b					b				