Fall 2023, Math 320: Week 7 Problem Set
Due: Tuesday, October 17th, 2023
Isomorphisms and Homomorphisms

Discussion problems. The problems below should be worked on in class.

(D1) *Homomorphisms.* The goal of this problem is to get practice with homomorphisms.

(a) Determine whether each of the following maps is a homomorphism.

(i) $\varphi : \mathbb{Z} \to \mathbb{Z}$ given by $\varphi(a) = a + 3$.

(ii) $\varphi : \mathbb{Z} \to \mathbb{Z}$ given by $\varphi(a) = 2a$.

(iii) $\varphi : \mathbb{Z} \to M_2(\mathbb{R})$ given by $\varphi(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

(iv) $\varphi : R \to S$ given by $\varphi\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$, where $R, S \subseteq M_2(\mathbb{R})$ are the set of all upper triangular and lower triangular matrices, respectively.

(b) Consider the map $\varphi : \mathbb{R}[x] \to \mathbb{R}$ given by

$$\varphi(a_0 + a_1x + a_2x^2 + \cdots + a_dx^d) = a_0$$

for $d \geq 0$ and $a_0, a_1, \ldots, a_d \in \mathbb{R}$. Prove φ is a homomorphism, but not an isomorphism.

(D2) *Specifying a homomorphism with minimal information.* Suppose $\varphi : \mathbb{R}[x] \to \mathbb{R}[y]$ is an unspecified homomorphism. Suppose that $\varphi(x) = y+1$, but **make no other assumptions** (for instance, we aren't given what $\varphi(x^2)$ is).

(a) Find $\varphi(x^3)$, $\varphi(x^2 + 2x)$, and $\varphi(x^5 + 9x^3 + 91x)$. Argue that your answer for each is the only possibility.

(b) Fill in the details in the following proof that $\varphi(1) = 1$.

Proof. Since φ is a homomorphism, we have

$$\varphi(1) \cdot (y + 1) = \varphi(1) \cdot \varphi(x) = \underline{\hspace{3cm}} = \underline{\hspace{3cm}} = y + 1.$$

Since $y+1$ is a nonzero and ______________________, cancellation yields $\varphi(1) = 1$. ☐

(c) What must $\varphi(0)$ be? What about $\varphi(-x)$? Prove both of your claims.

(d) What must $\varphi(3)$ be? What about $\varphi(1/3)$? What about $\varphi(\sqrt{3})$?

(e) Conjecture how many possible maps $\varphi : \mathbb{R}[x] \to \mathbb{R}[y]$ with $\varphi(x) = y + 1$ exist (you do **not** need to prove it). For which constants a have we not determined what $\varphi(a)$ is?

(D3) *Constructing isomorphisms.* Prove each of the following isomorphisms.

(a) $\mathbb{Z}_3 \times \mathbb{Z}_4 \cong \mathbb{Z}_{12}$.
Hint: use the map $\varphi : \mathbb{Z}_{12} \to \mathbb{Z}_3 \times \mathbb{Z}_4$ given by $\varphi([a]_{12}) = ([a]_3, [a]_4)$. Be sure to prove φ is well-defined!

(b) $R \cong \mathbb{Z}$, where $R = \{(a, a) \in \mathbb{Z} \times \mathbb{Z} : a \in \mathbb{Z}\}$.

(c) $R \cong \mathbb{R} \times \mathbb{R}$, where

$$R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R} \right\} \subseteq M(\mathbb{R})$$

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Determine whether each of the following maps is a homomorphism.

(a) $\varphi : \mathbb{Q} \to \mathbb{Q}$ given by $\varphi(0) = 0$ and $\varphi(a) = \frac{1}{a}$ for all nonzero $a \in \mathbb{Q}$.

(b) $\varphi : M(\mathbb{R}) \to M(\mathbb{R})$ given by $\varphi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ for all $a, b, c, d \in \mathbb{R}$.

(c) $\varphi : \mathbb{Z}_6 \to \mathbb{Z}_{18}$ with $\varphi([a]_6) = [a]_{18}$ for all $a \in \mathbb{Z}$. Hint: check whether φ is well-defined!

(d) $\varphi : \mathbb{Z}_{18} \to \mathbb{Z}_6$ with $\varphi([a]_{18}) = [a]_6$ for all $a \in \mathbb{Z}$. Hint: check whether φ is well-defined!

(H2) Consider the ring $(R, \oplus, \odot)$, where $R = \mathbb{Z}$,

$$a \oplus b = a + b + 1 \qquad \text{and} \qquad a \odot b = ab + a + b.$$

Prove that $R \cong \mathbb{Z}$. You do **not** need to prove R is a ring, as you already did so on a previous homework assignment.