Fall 2023, Math 320: Week 12 Problem Set
Due: Thursday, November 30th, 2023
Quotient Rings

Discussion problems. The problems below should be worked on in class.

(D1) *Polynomial quotients over $\mathbb{Z}_n$.*

(a) Consider the ring $R = \mathbb{Z}_2[x]/\langle x^2 + 1\rangle$.
  (i) List every element of R. Identify 0_R and 1_R.
  (ii) Write down the operation tables for R.
  (iii) Is R an integral domain? Is R a field?

(b) Answer the same questions for the ring $T = \mathbb{Z}_2[x]/\langle x^2 + x + 1\rangle$. Is it possible $R \cong T$?

(c) Demonstrate $[x - 1] \in \mathbb{Z}_7[x]/\langle x^2 - 3x + 2\rangle$ is a zero-divisor.

(d) Demonstrate $[x^2 - 1] \in \mathbb{Z}_7[x]/\langle x^2 - 3x + 2\rangle$ is a zero-divisor.

(e) Demonstrate $[x + 1] \in \mathbb{Z}_3[x]/\langle x^2 - 2x + 1\rangle$ is a unit.

(f) Determine whether $[x^2 + 1] \in \mathbb{Z}_5[x]/\langle x^3 - 2x^2 + x\rangle$ is a zero-divisor, a unit, or neither.

(g) Determine whether $[x^2 + 4] \in \mathbb{Z}_5[x]/\langle x^3 - 2x^2 + x\rangle$ is a zero-divisor, a unit, or neither.

(h) Locate a field F with exactly 4 elements (hint: look at parts (a) and (b)). Additionally, locate a field with 8 elements, a field with 9 elements, and a field with 27 elements.

(D2) *Identifying familiar quotient rings.*

(a) Prove $\mathbb{Q}[\sqrt{5}] = \{a + b\sqrt{5} : a, b \in \mathbb{Q}\} \subseteq \mathbb{R}$ is a field by locating an inverse for every nonzero element.

(b) Find a field F and a polynomial $p(x)$ so that $\mathbb{Q}[\sqrt{5}] \cong F[x]/\langle p(x)\rangle$.

(c) Locate a degree-2 polynomial $f(x) \in \mathbb{Q}[x]$ that has $a = 1 + \sqrt{5}$ as a root.

(d) Consider the **set**

$$R = \{a + b\sqrt[3]{2} : a, b \in \mathbb{Q}\} \subseteq \mathbb{R}.$$

Is R a ring? If not, what must be added to ensure R is closed under both operations?

(e) Based on part (b), how should $\mathbb{Q}[\sqrt[3]{2}]$ be defined so that it is a subring of $\mathbb{R}$?

(f) Find a field F and a polynomial $p(x)$ so that $\mathbb{Q}[\sqrt[3]{2}] \cong F[x]/\langle p(x)\rangle$.

(g) Let $R = \mathbb{Q}[x]/\langle x^2 - 2x - 1\rangle$. Locate an element $a \in R$ such that $a^2 = [2]$.
Hint: $(x - 1)^2 - 2 = x^2 - 2x - 1$.

(h) Identify a subring of $\mathbb{R}$ isomorphic to the ring R from the previous part.
Hint: figure out which real number the element $[x] \in R$ behaves like.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) (a) Locate a polynomial $f(x) \in \mathbb{Q}[x]$ such that $f(\sqrt{2}+\sqrt{3}) = 0$.
Hint: this can be done so that $\deg f(x) = 4$. Begin with $a = \sqrt{2}+\sqrt{3}$ and manipulate the equation until no square roots remain.

(b) Based on part (a), fill in the blank so that

$$\mathbb{Q}[\sqrt{2}+\sqrt{3}] = \{\underline{\hspace{6cm}} : a, b, c, d \in \mathbb{Q}\}$$

is a subring of $\mathbb{R}$ containing $\mathbb{Q}$ and $\sqrt{2}+\sqrt{3}$.

(c) Demonstrate that $\sqrt{2} \in \mathbb{Q}[\sqrt{2}+\sqrt{3}]$ and $\sqrt{3} \in \mathbb{Q}[\sqrt{2}+\sqrt{3}]$.

(H2) Let $R = \mathbb{Z}_2[x]/\langle x^3+x+1\rangle$. Find a **specific** element $r \in R$ such that $R\backslash\{0\} = \{r, r^2, r^3, \ldots\}$ (that is, so that every nonzero element of R is a power of r).

Hint: start by listing all of the elements of R.

(H3) Let $R_1 = \mathbb{Z}_2[x]/\langle x^2\rangle$ and $R_2 = \mathbb{Z}_2[y]/\langle y^2+1\rangle$. Prove $R_1 \cong R_2$.

Hint: write out the addition and multiplication tables for each ring, as they can be used to find an isomorphism $\phi : R_1 \to R_2$ and to verify it is indeed an isomorphism.

(H4) Determine whether each of the following statements is true or false. Prove each true statement, and give a counterexample for each false statement.

(a) The ring $R = \mathbb{Z}_4[x]/\langle x^2+1\rangle$ contains no zero-divisors.
(b) The ring $R = \mathbb{Z}_5[x]/\langle x^2+1\rangle$ contains no zero-divisors.
(c) The ring $R = \mathbb{Z}_2[x]/\langle x^2+x+1\rangle$ is isomorphic to $\mathbb{Z}_4$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Let

$$F = \mathbb{Z}_2[x]/\langle x^3+x+1\rangle \qquad \text{and} \qquad F' = \mathbb{Z}_2[y]/\langle y^3+y^2+1\rangle.$$

Find an isomorphism $\varphi : F \to F'$.