Fall 2023, Math 320: Week 14 Problem Set
Due: Tuesday, December 5th, 2023
Introduction to Groups

Discussion problems. The problems below should be worked on in class.

(D1) *Checking group axioms.* Determine whether each of the following sets G forms a group under the given operation $*$.

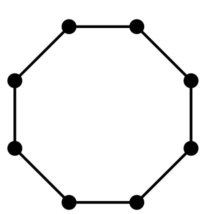
(a) $G = \mathbb{Z}$; $a * b = a \cdot b$.
(b) $G = \mathbb{Z}$; $a * b = a - b$.
(c) G is the set of nonzero rational numbers; $a * b = a/b$.
(d) $G = \mathbb{Z}_{\geq 0}$; $a * b = a + b$.
(e) $G = \mathbb{Z}_{\geq 1}$; $a * b = ab$.
(f) $G = \mathbb{Z}_{10}$; $a * b = ab$ (i.e. standard multiplication in $\mathbb{Z}_{10}$).
(g) $G = \{1, 3, 7, 9\} \subseteq \mathbb{Z}_{10}$; $a * b = ab$ (i.e. standard multiplication in $\mathbb{Z}_{10}$).
(h) $G = \{1, 3, 7, 9\} \subseteq \mathbb{Z}_9$; $a * b = ab$ (i.e. standard multiplication in $\mathbb{Z}_9$).
(i) $G = \{1, 2, 4, 5, 7, 8\} \subseteq \mathbb{Z}_9$; $a * b = ab$ (i.e. standard multiplication in $\mathbb{Z}_9$).
(j) $G = \mathbb{R} \times \mathbb{R}$; $(a, b) * (c, d) = (ac, bd)$.
(k) $G = \mathbb{R}^* \times \mathbb{R}$ where $\mathbb{R}^*$ is the set of nonzero real numbers; $(a, b) * (c, d) = (ac, ad + bc)$.

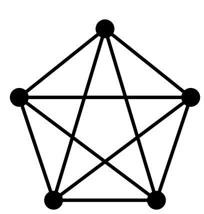
(D2) *Graph automorphisms.* Use the graphs at the bottom as a guide for this problem.

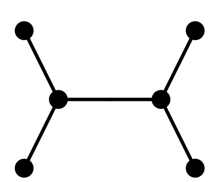
(a) Complete the following proof that for each $n \geq 3$, the n-vertex cycle graph $G = C_n$ has exactly $2n$ automorphisms, and thus $\mathbb{A}(C_n) \cong D_n$ as described in lecture.

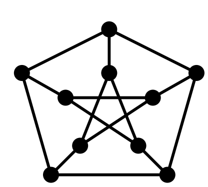
Proof. Label the vertices of G by $1, 2, \ldots, n \in \mathbb{Z}_n$ in a clockwise fashion.
Suppose $f : \mathbb{Z}_n \to \mathbb{Z}_n$ is an automorphism of G, and let $v = f(1)$. We must have $f(2) = v + 1$ or ____ since these are the only vertices connected to v. If $f(2) = v + 1$, then $f(3) =$ ____, $f(4) =$ ____, and so on since __________. Similarly, if $f(2) =$ ____, the remaining values of f must again follow in cyclic order. This means f is determined by choosing the value $f(1)$ (____ possibilities) and then an adjacent value for $f(2)$ (____ possibilities), yielding $2n$ total automorphisms. $\square$

(b) Prove that if $n \geq 2$ and $G = K_n$ is the complete graph, then **every** permutation of its vertices is an automorphism. Which group from lecture is $\mathbb{A}(K_n)$?
(c) Find (with proof!) all automorphisms of the "appendage graph" (depicted below). Is this isomorphic to a "familiar" group?
(d) Find as many automorphisms of the "Peterson graph" (depicted below) as you can.


Cycle graph C_8


Complete graph K_5


Appendage graph


Peterson graph

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Each of the following sets of 2×2 real matrices does **not** form a group under matrix multiplication. For each, locate an axiom that is violated, and give a specific example demonstrating this is the case.

(a) $M = M(\mathbb{R})$ (that is, the set of **all** 2×2 matrices with real entries).

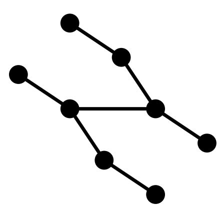
(b) $M = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \text{ with } a \neq 0 \text{ and } b \neq 0 \right\}$.

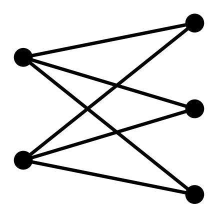
(H2) Determine whether each of the following sets G form a group under the given operation $*$. If yes, prove they form a group. If no, give a specific example demonstrating that one of the axioms is violated.

(a) G is the set of nonzero real numbers; $a * b = |a| \cdot b$.

(b) $G = \mathbb{R}$; $a * b = a + b + 3$.

(H3) For each of the following graphs, list its automorphisms (you may write each element as a permutation or in terms of other automorphisms like we did with the dihedral group) and identify a group from class that is isomorphic to its automorphism group.





Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Identify a graph H whose automorphism group $\mathbb{A}(H)$ is isomorphic to $(\mathbb{Z}_5, +)$.

Hint: to achieve this, the graph should have **exactly** 5 automorphisms, and each non-identity automorphism $\sigma \in \mathbb{A}(H)$ should have $\sigma^5 = e$.