**Fall 2026, Math 296: Week 1 Problem Set**
**Due: Thursday, September 3rd, 2026**
**Mathematical Exploration: Divisibility of Natural Numbers**

**Discussion problems.** The problems below should be worked on in class.

(D1) (a) Compare your answers to Preliminary Problem (P2). Come to a consensus within your group for a correct and well-written definition of "*root*."

(b) Explain why each of the following definitions of "prime natural number" is problematic. Then write a single, correct definition. Hint: remember that 1 is **not** prime!

(i) A natural number $n$ is *prime* if $n$ is not divisible by any other natural number.

(ii) A natural number $n$ is *prime* if $n$ cannot be written as a product of other primes.

(iii) A natural number $n$ is *prime* if the only natural numbers dividing $n$ are 1 and $n$.

(D2) (a) Consider the following statement.

"For all natural numbers $a$ and $b$, if $a \mid b$ and $b \mid a$, then $a = b$."

Fill in the blanks in the following proof of this statement.

*Proof.* Since ____, there exists a natural number $c$ such that $b = ac$, and since $b \mid a$, _____. Substitution yields

$$a = bd = _____.$$

Cancelling $a$ from both sides, we obtain _____. Since _____, we must have $c = d = 1$, and thus $a = b$. □

(b) Consider the following statement.

"For every integer $n$, we have $n^2 \geq n$."

The following proof of the statement has **2 errors**. Locate and correct both errors to obtain a valid proof.

*Proof.* We proceed by considering cases.

- If $n$ is negative, then $n^2 = n \cdot n$ is a product of two negative numbers, and thus is positive. So $n^2 > 0 > n$.
- If $n$ is positive, then $n^2 = n \cdot n > n \cdot 1 = n$.

In each case, we have shown $n^2 \geq n$. □

(D3) (a) Prove that for all natural numbers $a$ and $b$, if $a \mid b$, then $a \mid 7b$.

(b) Is there anything special about "7" in the above statement? State a more general result involving a third natural number $c$, and adjust your proof from part (a) accordingly.

(D4) Determine whether each of the following statements is true or false. Prove your claims.

Hint: remember that as mathematicians, when we say "all" we mean it literally!

(a) For all natural numbers $a, b$, and $c$, if $a \mid c$ and $b \mid c$, then $ab \mid c$.

(b) For all natural numbers $a, b$, and $c$, if $a \mid b$ and $b \mid c$, then $a \mid c$.

(c) For all natural numbers $a, b$, and $c$, if $a \mid b$ and $a \mid c$, then $a \mid (2b + 3c)$.

(d) For all natural numbers $a, b$, and $c$, if $a \mid b$ and $a \mid (b + c)$, then $a \mid c$.

**Homework problems.** You must submit *all* homework problems in order to receive full credit.

(H1) Suppose $x$ and $y$ are integers.

(a) Prove that if $y = x^2$ or $y = -x$, then $x^3 + x^2y = y^2 + xy$.

(b) Prove that if $x^3 + x^2y = y^2 + xy$, then $y = x^2$ or $y = -x$.

(H2) Prove that if $q$ is a natural number and $n = 5q + 2$, then $5 \mid (n^2 + 1)$.

(H3) Locate 5 consecutive integers, none of which are prime. Then locate 17 consecutive integers, none of which are prime.

(H4) Determine whether each of the following statements is true or false. Prove your claims.

(a) For every positive integer $n$, the number $n^2 + n + 41$ is prime.

(b) For all natural numbers $a, b$, and $c$, if $a \mid b$ and $a \mid (b + c)$, then $a \mid c$.

**Challenge problems.** Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Prove that if $p \geq 5$ and $q \geq 5$ are prime, then $24 \mid (p^2 - q^2)$.