**Fall 2026, Math 296: Week 2 Problem Set**
**Due: Thursday, September 10th, 2026**
**Mathematical Exploration: Digging Deeper**

**Discussion problems.** The problems below should be worked on in class.

(D1) Compare your answers to the Preliminary Problems.

(D2) (a) Fill in the blanks in the following proof that every natural number $n \geq 2$ can be factored as a product of primes.

*Proof.* Assume to the contrary that $n \geq 2$ is a natural number that _____. Assume $n$ is the smallest natural number with this property (that is, _____). This means $n$ is not _____, as otherwise $n = n$ would be a factorization. As such, $n = ab$ for some natural numbers $a$ and $b$ with _____. By the minimality of $n$, both $a$ and $b$ can be _____, that is,

$$a = p_1 \cdots p_k \qquad \text{and} \qquad b = q_1 \cdots q_\ell$$

for some primes $p_1, \ldots, p_k, q_1, \ldots, q_\ell$. This means

$$n = _____$$

which contradicts the assumption that $n$ _____. □

(b) Below is a "proof" that there are infinitely many primes. Locate and correct the error.

*Proof.* By way of contradiction, suppose there are only $k$ primes $p_1, \ldots, p_k$. Let

$$a = p_1 \cdots p_k + 2.$$

For each $i$, we have $p_i \mid (p_1 \cdots p_k)$, so $p_i \nmid a$. This means no prime numbers divide $a$, and thus $a$ cannot be written as a product of primes, which contradicts part (a). □

(D3) (a) Determine which of the numbers $1 + \sqrt{2}$, $2\sqrt{2}$, and $\sqrt{2} \cdot \sqrt{2}$ is rational.

(b) Place the following sentences in order to form a valid proof that $\log_2(3)$ is irrational.

(1) Let $\log_2(3) = m/n$ for some natural numbers $m$ and $n$.
(2) This is a contradiction, so our assumption $\log_2(3)$ is rational must be false.
(3) We then have $(2^{m/n})^n = 3^n$, which implies $2^m = 3^n$.
(4) Assume to the contrary that $\log_2(3)$ is rational.
(5) By the definition of $\log_2$, we have $2^{m/n} = 3$.
(6) However, since $m > 0$, we know $2^m$ is even, but $3^n$ is odd.

(c) State and prove a strictly more general result that gives conditions on natural numbers $a, b \geq 2$ that ensure $\log_a(b)$ is irrational. Try to make your result as general as you can, while minimizing the modifications to the proof from part (b).

(D4) Prove that for any integer $n$, there exists an integer $q$ such that $n^2 = 4q$ or $n^2 = 4q + 1$.

Hint: use cases based on the remainder upon dividing $n$ by 4 (via Division Algorithm).

(D5) Determine whether each of the following statements is true or false. Prove your claims.

(a) For any integer $n$, there exists an integer $q$ such that $n^2 = 5q$ or $n^2 = 5q + 1$.

(b) If $p > 2$ is prime, then $3p + 2$ is prime.

**Homework problems.** You must submit *all* homework problems in order to receive full credit.

(H1) What are the possible remainders a perfect square can have when divided by 7? Demonstrate that each of your chosen remainders is indeed possible, and state and prove a result analogous to Problem (D4) ensuring no other remainders are possible.

(H2) Prove that $\sqrt[3]{2}$ is irrational.

Hint: use, as a guide, the proof from class that $\sqrt{2}$ is irrational.

(H3) Complete Problem (D3)(c).

(H4) Determine whether each of the following statements is true or false. Prove your claims.

(a) For all natural numbers $a, b \geq 2$, if $\log_a(b)$ is rational, then $b = a^k$ for some integer $k$.

(b) For all natural numbers $a$, $b$, and $c$, if $a \mid bc$, then $a \mid b$ or $a \mid c$.

(c) For all primes $p > 3$, the number $p^2 + 2$ is composite.

**Challenge problems.** Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Prove there are infinitely many primes that each have remainder 2 when divided by 3.