Fall 2026, Math 296: Week 4 Problem Set
Due: Thursday, September 24th, 2026
Proof Methods and Quantifiers

Discussion problems. The problems below should be worked on in class.

(D1) Consider the following propositions from Problem (P1).

(i) $\forall x \in \mathbb{Z}, x > 0$ (ii) $\exists x \in \mathbb{R}, \forall y \in \mathbb{Q}, x \neq y$ (iii) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, y^2 = x$

(a) Compare your answers to Problem (P1). Then work together to find the most concise (and clear!) sentences possible. As an extra challenge: for the statements (i), (ii), and (iii), this can be done using only 4, 6, and 5 words, respectively!

(b) Write a brief proof/disproof for each statement. Use the proofs from lecture as a guide.

(c) Determine the negation of each statement. Write each as a full English sentence, as well as symbolically. Then, write a proof/disproof of each negated statement.

(D2) Suppose $p(x)$ is a statement involving the variable x, and that D is a set.

(a) Develop a one sentence explanation of why the following holds:

$$\sim(\forall x \in D, p(x)) \equiv \exists x \in D, \sim p(x).$$

Hint: you may find it helpful to first use Problem (D1)(i) as an example.

(b) Find a proposition that is equivalent to $\sim(\exists x \in D, p(x))$ and involves $\sim p(x)$.

(c) Negate the proposition

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{Z}, xy = 0.$$

Determine which of the proposition and its negation is true, and prove/disprove each.

(d) Consider the following propositions.

(i) $\forall x \in \mathbb{R}, \exists y \in \mathbb{Z}, x > y$ (ii) $\exists y \in \mathbb{Z}, \forall x \in \mathbb{R}, x > y$

Do these have the same truth value? What does this tell you about quantifier order?

(e) Consider the following propositions.

(i) $\forall x \in \mathbb{R}, \exists y \in \mathbb{Z}, xy \in \mathbb{Z}$ (ii) $\exists y \in \mathbb{Z}, \forall x \in \mathbb{R}, xy \in \mathbb{Z}$

Do these have the same truth value? Give a proof/disproof of each.

(D3) Consider the following statement.

"If $a + b$ and ab are both even, then a and b are both even."

Locate the error in the following proof of the statement. Then write a correct proof.

Proof. Suppose it is **not** the case that a and b are both even. This means a and b are both odd. Necessarily, $a + b$ is even, but ab is odd. As such, $a + b$ and ab are not both even. This completes the proof. □

(D4) Consider the following statement.

"For all $x \in \mathbb{R}$, if x^2 is irrational, then $3x$ is irrational."

(a) Fill in the blanks in the following proof of the above statement.

Proof. Fix ____, and suppose ____ is ____. Towards a contradiction, assume ____ is ____. There exist $a, b \in \mathbb{Z}$ with $b \neq 0$ such that ____ = ____. Substituting, we see

____ = _____ = _____.

Since _____ $\neq 0$, we conclude _____ is _____. This contradicts our assumption that _____ is _____. □

(b) Using a different color, edit your completed proof to turn it into a contrapositive proof. Note: this can be done with no additions, only deletions!

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) (a) Give a direct proof that if $x \in \mathbb{Z}$ is odd, then $4 \mid 2x + 2$.

(b) Give a contrapositive proof of the statement in part (a).
Hint: use the division algorithm and consider cases based on the possible remainders.

(H2) Prove that x is rational if and only if $3x$ is rational.

(H3) Determine which of the following statements is true. Give a proof/disproof of each.

(a) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \forall z \in \mathbb{R}, x + y < z$.

(b) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \exists z \in \mathbb{R}, x + y < z$.

(c) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \forall z \in \mathbb{R}, x + y < z$.

(d) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, \forall z \in \mathbb{R}, x + y < z$.

(e) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, \exists z \in \mathbb{R}, x + y < z$.

(H4) Determine whether each of the following statements is true or false. Prove your claims.

(a) There exists $q \in \mathbb{Q}$ with $\sqrt{q} \in \mathbb{Q}$.

(b) For every $n \in \mathbb{Z}$, $n^2 > n$.

(c) There exist $a, b \in \mathbb{Z}$ with $956a - 108b = 157$.

(d) There exist unique positive integers a and b with $2a + 3b = 11$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Determine whether or not the following proposition is true, and give a proof/disproof.

$$\exists! x \in \mathbb{R}, \exists! y \in \mathbb{R}, y^2 = x$$