Fall 2026, Math 296: Week 5 Problem Set
Due: Thursday, October 1st, 2026
Induction

Discussion problems. The problems below should be worked on in class.

(D1) (a) Fill in the blanks in the following proof that for every $n \geq 1$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Proof. We proceed by induction on n.
Base case: $n = 1$. We obtain $1^2 =$ ____ $= \frac{1(2)(3)}{6}$, as expected.
Inductive step: assuming $n \geq 1$ and

$$_____ = \frac{n(n+1)(2n+1)}{6},$$

we wish to show

$$1^2 + 2^2 + \cdots + n^2 + (n+1)^2 = _____.$$

To this end, we see that

$$\begin{aligned} 1^2 + 2^2 + \cdots + n^2 + (n+1)^2 &= _____ \\ &= _____ \end{aligned}$$

as desired. □

(b) Rewrite all centered algebra expressions using sigma sums and avoiding "$\cdots$".

(D2) Suppose the sequence $a_0, a_1, a_2, \ldots$ satisfies $a_0 = 1$, $a_1 = 2$, and

$$a_n = 2a_{n-1} - a_{n-2}$$

for $n \geq 2$. Find a formula for a_n by experimentation, then prove it using induction. Be sure to clearly state what you are proving (e.g., as a self-contained theorem alongside its proof).

(D3) *Strong induction.* To prove "for all $n \in \mathbb{Z}$ with $n \geq 1$, $p(n)$ holds" it suffices to:

- prove $p(1)$; and
- prove for all $n \geq 1$, $p(1) \wedge p(2) \wedge \cdots \wedge p(n) \Rightarrow p(n+1)$.

(a) Explain why $p(1)$, $p(2)$, $p(3)$, and $p(4)$ all follow as conclusions from strong induction. Hint: first write the base case and the first few implications from the inductive step.

(b) Suppose the sequence $a_0, a_1, a_2, \ldots$ satisfies $a_0 = 0$ and

$$a_n = a_0 + a_1 + \cdots + a_{n-1} \qquad \text{whenever} \qquad n \geq 1.$$

Prove $a_n = 2^n - 1$ for all $n \geq 0$ using a combination of strong induction and reindexed induction, and be **extra careful** when setting up the inductive step.

(D4) Use induction on n to prove that

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$$

for all $n \geq 1$. Write your proof **without** introducing any new variables (i.e., just using n), and be careful when stating your inductive hypothesis and setting up the inductive step!

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Prove that

$$\sum_{k=1}^{n}(-1)^k k^2 = \tfrac{1}{2}n(n+1)(-1)^n$$

for all $n \in \mathbb{Z}$ with $n \geq 1$.

Hint: use sigma sums rather than "$\cdots$"; it helps prevent lost signs in alternating sums.

(H2) Suppose $x \in \mathbb{R}$ with $x > 0$.

(a) Use induction on n to prove that

$$(1+x)^n \geq 1 + nx$$

for all $n \in \mathbb{Z}$ with $n \geq 1$.

Note: you may **not** use the binomial theorem in your proof.

(b) Is part (a) still true if the "$x > 0$" assumption is omitted? If not, where in your proof did you use this assumption?

(H3) (a) Prove that for every $n \in \mathbb{Z}$ with $n \geq 4$, we have $n! \geq n^2$.
Hint: recall that $n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$.

(b) Why is the requirement "$n \geq 4$" present in part (a), and how was this reflected in your proof?

(H4) Suppose the sequence $a_0, a_1, a_2, \ldots$ satisfies $a_0 = 2$, $a_1 = 1$, and

$$a_n = a_{n-1} + 2a_{n-2}$$

for all $n \geq 2$.

(a) Conjecture a formula for a_n in terms of n.
Hint: your formula for a_n will likely have cases based on whether n is even or odd.

(b) Use induction on n to prove your conjecture in part (a).