Fall 2026, Math 296: Week 6 Problem Set
Due: Thursday, October 8th, 2026
A Few More Proof Methods

Discussion problems. The problems below should be worked on in class.

(D1) Compare answers to the preliminary problems, and come to a consensus within your group.

(D2) Consider the following statement.

"For all $x \in \mathbb{R}$, if x^2 is irrational, then $3x$ is irrational."

(a) Fill in the blanks in the following proof of the above statement.

Proof. Fix ____, and suppose ____ is ____. Towards a contradiction, assume ____ is ____. There exist $a, b \in \mathbb{Z}$ with $b \neq 0$ such that ____ = ____. Substituting, we see

____ = _____ = _____.

Since _____ $\neq 0$, we conclude _____ is _____. This contradicts our assumption that _____ is _____. □

(b) Using a different color, edit your completed proof to turn it into a contrapositive proof. Note: this can be done with no additions, only deletions!

(D3) Consider the following statement.

"If $a + b$ and ab are both even, then a and b are both even."

Locate the error in the following proof of the statement. Then write a correct proof.

Proof. Suppose it is **not** the case that a and b are both even. This means a and b are both odd. Necessarily, $a + b$ is even, but ab is odd. As such, $a + b$ and ab are not both even. This completes the proof. □

(D4) Fill in the blanks in the following proof that for all $n \in \mathbb{N}$, $5^n > 2^{n+1}$.

Proof. Let S denote the collection of all $n \in \mathbb{N}$ for which _____. If S is empty, we are done. As such, by way of contradiction, suppose _____. Since $1 \leq n$ for every $n \in S$, by the _____ we know S has a _____ element $m \in S$. In what follows, we will show $m - 1 \in S$. Since $5^1 = 5 > 4 = 2^{1+1}$, we know $m \neq$ _____, so since $m \in \mathbb{N}$, we have $m \geq$ _____ and thus $m - 1 \in \mathbb{N}$. Moreover,

$$5^{m-1} = \tfrac{1}{5}5^m \leq \tfrac{1}{5}2^{m+1} \leq \tfrac{1}{2}2^{m+1} = 2^m,$$

so we conclude _____ $\in S$. This contradicts _____. □

(D5) (a) Below is a proof that if $a \in \mathbb{Z}_{\geq 0}$, then $a = 7q + r$ for some $q, r \in \mathbb{Z}$ with $0 \leq r < 7$. The proof below uses induction on a, but contains an error. Locate/correct the error.

Proof. We proceed by induction on a.
Base cases: suppose $a = 0, 1, \ldots, 5$, or 6. Choosing $q = 0$ and $r = a$, we see $7q + r = a$.
Inductive step: fix $a \geq 7$, and suppose there exist $q', r' \in \mathbb{Z}$ with $0 \leq r' < 7$ such that

$$a - 7 = 7q' + r'.$$

Rearranging yields

$$a = 7(q' + 1) + r',$$

and choosing $q = q' + 1$ and $r = r' + 1$ completes the proof. □

(b) Adjust the above proof to prove that for any $a, b \in \mathbb{Z}_{\geq 0}$ with $b \geq 1$, there exist $q, r \in \mathbb{Z}$ with $0 \leq r < b$ such that $a = qb + r$.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Consider the Fibonacci sequence $F_0, F_1, F_2, \ldots$ defined by $F_0 = 0$, $F_1 = 1$, and

$$F_n = F_{n-1} + F_{n-2}$$

for each $n \geq 2$. Prove that for each $n \in \mathbb{Z}$ with $n \geq 11$, we have

$$1.5^n \leq F_n \leq 2^n.$$

(H2) Fix $n \in \mathbb{Z}$. Prove that the following are equivalent:

(i) n is even;

(ii) $\frac{1}{2}n(n+2)$ is an integer; and

(iii) $\frac{1}{2}n(n+2)$ is even.

(H3) Using as a model the proof of the division algorithm from the lecture in week 2, prove the following modified version of the division algorithm: for any $a, b \in \mathbb{Z}$ with $b \geq 1$, there exist unique $q, r \in \mathbb{Z}$ with $1 \leq r \leq b$ such that $a = qb + r$.

(H4) Determine whether each of the following statements is true or false. Prove your claims.

(a) For all $x \in \mathbb{Z}$, $|5x + 8| \geq 2$.

(b) For all $x \in \mathbb{Z}$, $|2x - 7| + 2x \geq 0$.