Fall 2026, Math 296: Week 9 Problem Set
Due: Thursday, November 5th, 2026
Functions

Discussion problems. The problems below should be worked on in class.

(D1) Compare answers to the preliminary problems, and come to a consensus within your group.

(D2) Determine which of the following are injective and/or surjective, and find the image of each.

(a) $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 2x - 6$.
(b) $f : \mathbb{Z} \to \mathbb{N}$, $f(x) = |x - 2| + 1$.
(c) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, $f(a, b) = 2a + 3b$.
(d) $f : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, $f(a) = (a, 0)$.

(D3) Suppose $f : A \to B$ and $g : B \to C$, and let $h : A \to C$ denote their composition $g \circ f$.

(a) Prove that if f and g are both injective, then h is injective.
(b) Prove that if f and g are both surjective, then h is surjective.

(D4) Suppose $f : A \to B$. We say $g : B \to A$ is a *left inverse* of f if $g(f(a)) = a$ for all $a \in A$.

(a) Define $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ by

$$f(a) = 2a \qquad \text{and} \qquad g(b) = \begin{cases} \frac{1}{2}b & \text{if } b \text{ is even;} \\ b - 1 & \text{if } b \text{ is odd.} \end{cases}$$

Prove that g is a left inverse of f. Then locate a **different** left inverse of f.
(b) Define $f : \{1, 2\} \to \{1, 2, 3, 4\}$ by $f(a) = a + 1$. How many left inverses does f have?
(c) Prove the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(a) = a^2$ does not have a left inverse.
(d) Suppose $f : A \to B$ is injective. Briefly explain how you could construct a left inverse $g : B \to A$ of f (hint: decide where g should send each element $b \in B$). Where do you use the fact that f is injective? Are there any requirements on the sets A or B?
(e) Prove that if $f : A \to B$ has a left inverse, then f is injective.

(D5) Suppose $f : A \to B$. We say $g : B \to A$ is a *right inverse* of f if $f(g(b)) = b$ for all $b \in B$.

(a) Recall $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Define $f : \mathbb{Z} \to \mathbb{N}_0$ and $g : \mathbb{N}_0 \to \mathbb{Z}$ by

$$f(a) = \begin{cases} a & \text{if } a \geq 0; \\ -a & \text{if } a < 0, \end{cases} \qquad \text{and} \qquad g(b) = b.$$

Prove that g is a right inverse of f. Then locate a **different** right inverse of f.
(b) Define $f : \{-1, 0, 1\} \to \{0, 1\}$ by $f(a) = |a|$. How many right inverses does f have?
(c) Prove the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(a) = a^2$ does not have a right inverse.
(d) Suppose $f : A \to B$ is surjective. Briefly explain how you could construct a right inverse $g : B \to A$ of f. Where do you use the fact that f is surjective? Are there any requirements on the sets A or B?
(e) Prove that if $f : A \to B$ has a right inverse, then f is surjective.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) (a) Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ given by

$$f(a) = \begin{cases} a - 2 & \text{if } a \text{ is even;} \\ 2a & \text{if } a \text{ is odd;} \end{cases}$$

Find $\mathrm{Im}(f)$.

(b) Locate a function $f : \mathbb{Z} \to \mathbb{Z}$ such that $\mathbb{Z} \setminus \mathrm{Im}(f) = \{4k : k \in \mathbb{Z}\}$.

(H2) Suppose $m, b \in \mathbb{Z}$ and define $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = mx + b$.

(a) Under what conditions on a and b is f injective? State your characterization as an "if and only if" statement, and prove it.

(b) Under what conditions on a and b is f surjective? State your characterization as an "if and only if" statement, and prove it.

(c) Under what conditions on a and b is f bijective? State your characterization as an "if and only if" statement, and prove it.

(H3) Revisit Problem (H2), replacing each instance of "$\mathbb{Z}$" with "$\mathbb{R}$" throughout.

(H4) Recall $\mathbb{N} = \{1, 2, 3, \ldots\}$. Locate a bijection $f : \mathbb{Z} \to \mathbb{N}$ such that the image of any positive number is even.

(H5) Suppose $f : A \to B$ and $g : B \to C$, and let $h : A \to C$ denote their composition $g \circ f$. Determine whether each of the following statements is true or false. Prove your claims.

(a) If h is injective, then f is injective.

(b) If h is injective, then g is injective.

(c) If h is surjective, then f is surjective.

(d) If h is surjective, then g is surjective.