Spring 2016, Math 485
Week 8 Homework

Prove one direction of the equivalence of any of the following matroid definitions. Remember: you will need to describe the method of transforming your matroid "form" into the other. For example, if $\mathcal{I}$ is the set of independent sets of a matroid, then the collection of bases $\mathcal{B}$ consists of the maximal (w.r.t. containment) sets in $\mathcal{I}$.

(Q1) Fix a finite set E and a collection of subsets $\mathcal{I} \subset 2^E$. Then $(E, \mathcal{I})$ is an *independence matroid* if

- (I1) $\emptyset \in \mathcal{I}$.
- (I2) If $I \in \mathcal{I}$ and $J \subset I$, then $J \in \mathcal{I}$.
- (I3) If $I, J \in \mathcal{I}$, $|J| < |I|$, then $\exists a \in I - J$ such that $J \cup a \in \mathcal{I}$.

(Q2) Fix a finite set E and a collection of subsets $\mathcal{B} \subset 2^E$. Then $(E, \mathcal{B})$ is a *basis matroid* if

- (B1) $\mathcal{B} \neq \emptyset$.
- (B2) For all $A, B \in \mathcal{B}$ and $b \in B - A$, $\exists a \in A - B$ such that $B - b \cup a \in \mathcal{B}$.

(Q3) Fix a finite set E and a collection of subsets $\mathcal{C} \subset 2^E$. Then $(E, \mathcal{C})$ is a *circuit matroid* if

- (C1) $\emptyset \neq \mathcal{C}$.
- (C2) $D \in \mathcal{C}$, $C \subsetneq D$, $C \notin \mathcal{C}$.
- (C3) If $C_1, C_2 \in \mathcal{C}$ distinct, $e \in C_1 \cap C_2$, then $\exists C_3 \in \mathcal{C}$ with $C_3 \subset (C_1 \cup C_2) - e$.

(Q4) Fix a finite set E and a function $r : 2^E \to \mathbb{Z}$. Then (E, r) is a *rank matroid* if

- (R1) $0 \leq r(X) \leq |X|$ $\forall X \subset E$.
- (R2) If $X \subset Y \subset E$, then $r(X) \leq r(Y)$.
- (R3) If $X, Y \subset E$, then
$$r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$$