Spring 2019, Math 320: Problem Set 5
Due: Tuesday, March 5th, 2019
Properties of Rings

Discussion problems. The problems below should be worked on in class.

(D1) *The ring structure of $\mathbb{Z}_n$.* The goal of this problem is to determine which elements of $\mathbb{Z}_n$ are zero-divisors, which are units, and which are neither.

  (a) Compare your answers to problem (P1). Then, find all zero divisors in $\mathbb{Z}_4$, $\mathbb{Z}_5$, and $\mathbb{Z}_6$.

  (b) Prove that if $m, n \geq 2$, then $\mathbb{Z}_{mn}$ is not an integral domain.

  (c) Suppose p is prime. Complete the following proof that $\mathbb{Z}_p$ is an integral domain. Hint: use the fact that if p is prime and $p \mid ab$, then $p \mid a$ or $p \mid b$.

  Proof. Fix $[a]_p, [b]_p \in \mathbb{Z}_p$, and suppose $[a]_p[b]_p = [0]_p$. We need to prove that either $[a]_p = [0]_p$ or $[b]_p = [0]_p$. Since $[0]_p = [a]_p[b]_p = [ab]_p$, we have $p \mid ab$. As such, ... $\square$

  (d) Multiply each element of $\mathbb{Z}_7$ by $[4]_7$ (i.e. find $[0]_7 \cdot [4]_7$, then $[1]_7 \cdot [4]_7$, and so forth). Do the same with $[5]_7$. What do you notice about which elements of $\mathbb{Z}_7$ appear?

  (e) Multiply every element of $\mathbb{Z}_{11}$ by $[3]_{11}$. Which elements of $\mathbb{Z}_{11}$ are obtained? Hint: you may want to "divide and conquer" within your group!

  (f) Suppose p is prime. Find and correct the error in the following proof that $\mathbb{Z}_p$ is a field.

  Proof. Fix an arbitrary $a \in \mathbb{Z}_p$. Since $\mathbb{Z}_p$ is finite, let $a_1, a_2, \ldots, a_p$ denote the complete list of distinct elements of $\mathbb{Z}_p$. We must find k so that $a_k \cdot a = [1]_p$. Consider the list

  $$a_1 \cdot a, \quad a_2 \cdot a, \quad \ldots, \quad a_p \cdot a.$$

  We claim these elements are all distinct (indeed, since $\mathbb{Z}_p$ is an integral domain, if $a_i \cdot a = a_j \cdot a$, then cancelling the a's yields $a_i = a_j$). This means every element of $\mathbb{Z}_p$ appears somewhere in the list. In particular, $[1]_p$ appears somewhere in the list, meaning for some k, we have $a_k \cdot a = [1]_p$. $\square$

  (g) Look carefully at the proof in part (f). What properties of $\mathbb{Z}_p$ were used in the proof? Use this to complete the following (much more general) result.

  Theorem (Theorem 3.9)**.** *If R is a integral domain and _____, then R is a field.*

  (h) Combining the results above, characterize (in terms of n) when $\mathbb{Z}_n$ is a field, when $\mathbb{Z}_n$ is an integral domain but not a field, and when $\mathbb{Z}_n$ is neither an integral domain nor a field. State your characterization formally (as a theorem), and **box it 3 times**.

(D2) *Cartesian products.*

  (a) Determine which elements of $\mathbb{Z}_5 \times \mathbb{Z}_4$ are units, and which are zero-divisors.

  (b) Suppose $m, n \geq 2$. Determine the units and zero-divisors of $\mathbb{Z}_m \times \mathbb{Z}_n$.

  (c) Suppose R_1 and R_2 are rings. Can there be elements of $R_1 \times R_2$ that are neither units nor zero-divisors?

  (d) Suppose R_1 and R_2 are rings. Determine which elements of $R_1 \times R_2$ are units, in terms of the units of R_1 and the units of R_2.

  (e) Suppose R_1 and R_2 are rings. Determine which elements of $R_1 \times R_2$ are zero-divisors, in terms of the zero-divisors of R_1 and the zero-divisors of R_2.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) *****EXCISED*****

(H2) Give an example of a ring in which the set of non-unit elements is closed under both addition and multiplication.

(H3) Suppose R is a ring and $S, T \subset R$ are subrings. Prove $S \cap T$ is a subring of R.

(H4) Determine whether each of the following statements is true or false. Prove each true statement, and give a counterexample for each false statement.

    (a) If R is a commutative ring and $a, b \in R$ are units, then ab is a unit.

    (b) If R is a commutative ring and $a, b \in R$ are zero divisors, then ab is a zero divisor. Hint: this one is tricky!

    (c) If R is a ring and $S, T \subset R$ are subrings, then $S \cup T$ is a subring of R.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Determine for which $m \geq 2$ the set of non-unit elements of $\mathbb{Z}_m$ is closed under both addition and multiplication. Prove your claim.