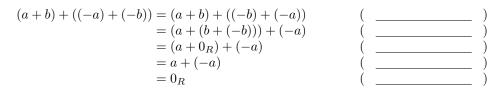
## Spring 2019, Math 320: Problem Set 6 Due: Tuesday, March 12th, 2019 Arithmetic in Rings

Discussion problems. The problems below should be worked on in class.

- (D1) Ring arithmetic. Suppose  $(R, +, \cdot)$  is a ring. Try to use only one axiom (or theorem) in each proof step in this problem.
  - (a) Fill in the justifications in the proof that for every  $a, b \in R$ , -(a+b) = (-a) + (-b).

*Proof.* By definition, -(a+b) is the additive inverse of a+b. To prove that (-a)+(-b) is the same element of R, we must prove that adding it to a+b yields 0. Indeed,



and a similar argument shows  $((-a) + (-b)) + (a + b) = 0_R$ .

- (b) Prove that for every  $a \in R$ , -(-a) = a.
- (c) Prove that for every  $a, b \in R$ , -(a b) = (-a) + b. Hint: a b means a + (-b).
- (d) Prove or disprove: if  $a, b \in R$  are units, then a + b is a unit.
- (e) Prove that if R has a multiplicative identity  $1_R \in R$ , then for every  $a \in R$ , (-1)a = -a.
- (f) Prove that if  $a, b \in R$  with a and ab both units, then b is a unit.
- (g) Locate a matrix J in the ring

$$S = \left\{ \begin{pmatrix} a & a \\ b & b \end{pmatrix} : a, b, \in \mathbb{R} \right\} \subset M(\mathbb{R})$$

that is a *right multiplicative identity* (that is, AJ = A for every  $A \in S$ ).

(D2) Identifying familiar rings in disguise. Throughout this problem, let  $R_1 = \mathbb{C}$  and

$$R_2 = \left\{ \left( \begin{array}{cc} a & b \\ -b & a \end{array} \right) : a, b \in \mathbb{R} \right\}.$$

- (a) Find the sum and product of 3 + 4i,  $5 + 6i \in R_1$ .
- (b) Find the sum and product of  $\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$ ,  $\begin{pmatrix} 5 & 6 \\ -6 & 5 \end{pmatrix} \in R_2$ . What do you notice?
- (c) Find the sum and product of two arbitrary elements  $a + bi, a' + b'i \in R_1$ .
- (d) Find the sum and product of two arbitrary elements of  $R_2$ , labeled analogously.
- (e) Is there a natural way to "match" each element of  $R_1$  with an element of  $R_2$ ? Define a function  $\varphi: R_1 \to R_2$  for this "matching".
- (f) Using your observations above, find an equation relating  $\varphi(c_1)$ ,  $\varphi(c_2)$ , and  $\varphi(c_1 + c_2)$  for  $c_1, c_2 \in R_1$ . Hint: start with  $c_1 = 3 + 4i$  and  $c_2 = 5 + 6i$  as a guide.

Homework problems. You must submit *all* homework problems in order to receive full credit.

- (H1) Suppose  $(R, +, \cdot)$  is a ring. Prove each of the following statements. Identify each ring axiom you use, and try to only use one axiom (or theorem) in each step.
  - (a) For any  $a, b, c, d \in R$ , we have a b + c = d if and only if a = b c + d.
  - (b) If  $a, b, c \in R$  with ab = 1 and ca = 1, then b = c.
  - (c) If R has unity and 1 = 0, then  $R = \{0\}$ .
- (H2) Fix a ring R and a unit  $a \in R$ . Prove by induction that  $(a^{-1})^n = (a^n)^{-1}$  for every  $n \in \mathbb{Z}_{>1}$ .
- (H3) Determine whether each of the following statements is true or false. Prove each true statement, and give a counterexample for each false statement.
  - (a) If R is a ring, then every element of R is either a unit or a zero-divisor.
  - (b) If  $F_1$  and  $F_2$  are fields, then  $F_1 \times F_2$  is a field.

**Challenge problems.** Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) An element a of a commutative ring R is *nilpotent* if  $a^k = 0_R$  for some  $k \ge 1$  (that is, a is nilpotent if some power of a equals  $0_R$ ). Prove that the set N of nilpotent elements of a commutative ring R is a subring of R.