Spring 2019, Math 320: Problem Set 8
Due: Tuesday, March 26th, 2019
Catch-up Week

Discussion problems. The problems below should be worked on in class.

(D1) *Review Problems.* The goal of this problem is to review content covered up to this point. Within your group, have each person vote on **2 parts** of this problem to work though together. Work through the parts in order of popularity.

(a) Locate a matrix J in the ring

$$S = \left\{ \begin{pmatrix} a & a \\ b & b \end{pmatrix} \in M(\mathbb{R}) : a, b, \in \mathbb{R} \right\}$$

that is a *right multiplicative identity* (that is, $AJ = A$ for every $A \in S$).

(b) Prove or disprove: for all $a, b, c \in \mathbb{Z}$ and $n \geq 2$, if $[a]_n[b]_n = [a]_n[c]_n$, then $[b]_n = [c]_n$.

(c) Prove that if $2^p - 1$ is prime, then p is prime. Hint: prove the contrapositive.

(d) Using only **one axiom at a time**, prove that if R is a ring with elements $a, b, c, d \in R$ satisfying $a - b = c - d$, then $a + d = b + c$.

(e) Fix a homomorphism $\varphi : R \to S$. Prove that the image of φ, that is,

$$\operatorname{Im}(\varphi) = \{\phi(r) : r \in R\},$$

is a subring of S.

(D2) *Constructing isomorphisms.* The goal of this problem is to get practice with isomorphisms.

(a) Prove each of the following.

(i) $\phi : \mathbb{Z}_{12} \to \mathbb{Z}_3 \times \mathbb{Z}_4$ given by $\phi([a]_{12}) = ([a]_3, [a]_4)$ is an isomorphism. Hint: be sure to prove ϕ is well-defined!

(ii) $\mathbb{R} \times \mathbb{R} \cong R$, where

$$R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R} \right\} \subset M(\mathbb{R})$$

(iii) $\mathbb{Z} \cong (R, \oplus, \odot)$, where $R = \mathbb{Z}$,

$$a \oplus b = a + b + 1 \qquad \text{and} \qquad a \odot b = ab + a + b.$$

for all $a, b \in R$. Hint: find 0_R and 1_R.

(iv) $\mathbb{Z} \cong R$, where $R = \{(a, a) : a \in \mathbb{Z}\} \subset \mathbb{Z} \times \mathbb{Z}$.

(b) Determine which of the following statements are true, and prove your assertions.

(i) $\mathbb{Z} \cong 2\mathbb{Z}$, where $2\mathbb{Z}$ denotes the set of even integers.

(ii) $\mathbb{Z}_{19} \cong \mathbb{Z}_{17}$.

(iii) $\mathbb{Z}_6$ is isomorphic to **some** subring of $M(\mathbb{R})$.

(iv) $\mathbb{Z}_3$ is isomorphic to **some** subring of $\mathbb{Z}_6$.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Suppose p is prime. Prove if $a^2 \equiv b^2 \bmod p$, then either $a \equiv b \bmod p$ or $a \equiv -b \bmod p$.

(H2) Is

$$R = \left\{ \begin{pmatrix} a & a \\ 0 & a \end{pmatrix} : a \in \mathbb{R} \right\} \subset M(\mathbb{R})$$

a subring of $M(\mathbb{R})$?

(H3) Consider the set $T = 2\mathbb{Z}$ of even integers, and operations $\oplus$ and $\odot$ given by

$$a \oplus b = a + b \qquad \text{and} \qquad a \odot b = ab/2$$

for $a, b \in R$.

(a) Locate the additive and multiplicative identities of T.

(b) Prove that $(T, \oplus, \odot)$ is closed under multiplication, satisfies the distributivity axiom, and that every element of T has an additive inverse.

(H4) Suppose R is a ring and $a \in R$ is nonzero and **not** a zero-divisor. Prove that if $ab = ac$, then $b = c$. Note: in this problem, use only one **axiom** in each step (no theorems!).

(H5) Let $(T, \oplus, \odot)$ denote the ring from Problem (H3) above (you may assume in this problem that T is a ring). Prove $T \cong (\mathbb{Z}, +, \cdot)$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Determine whether $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \cong M(\mathbb{R})$.