Spring 2019, Math 320: Week 13 Problem Set
Due: Tuesday, May 7th, 2019
Introduction to Groups

Discussion problems. The problems below should be completed in class.

(D1) *Checking group axioms.* Determine whether each of the following sets G forms a group under the given operation $*$.

(a) $G = \mathbb{Z}$; $a * b = a \cdot b$.
(b) $G = \mathbb{Z}$; $a * b = a - b$.
(c) G is the set of nonzero rational numbers; $a * b = a/b$.
(d) $G = \mathbb{Z}_{\geq 0}$; $a * b = a + b$.
(e) $G = \mathbb{Z}_{\geq 1}$; $a * b = ab$.
(f) $G = \mathbb{Z}_{10}$; $a * b = ab$ (i.e. standard multiplication in $\mathbb{Z}_{10}$).
(g) $G = \{1, 3, 7, 9\} \subset \mathbb{Z}_{10}$; $a * b = ab$ (i.e. standard multiplication in $\mathbb{Z}_{10}$).
(h) $G = \{1, 3, 7, 9\} \subset \mathbb{Z}_9$; $a * b = ab$ (i.e. standard multiplication in $\mathbb{Z}_9$).
(i) $G = \{1, 2, 4, 5, 7, 8\} \subset \mathbb{Z}_9$; $a * b = ab$ (i.e. standard multiplication in $\mathbb{Z}_9$).
(j) $G = \left\{x, 1-x, \dfrac{1}{x}, \dfrac{1}{1-x}, \dfrac{x}{1-x}, \dfrac{1-x}{x}\right\}$; $f(x) * g(x) = f(g(x))$.
(k) G is the set of functions $\mathbb{R} \to \mathbb{R}$ of the form $f(x) = ax + b$ with $a, b \in \mathbb{R}$ and $a \neq 0$ (that is, $f(x) = 2x + 3$ and $f(x) = 5x$ lie in G, but not $f(x) = x^2 + 2$ and $f(x) = 0$); $f(x) * g(x) = f(g(x))$.
(l) $G = \mathbb{R} \times \mathbb{R}$; $(a, b) * (c, d) = (ac, bd)$.
(m) $G = \mathbb{R}^* \times \mathbb{R}$ where $\mathbb{R}^*$ denotes the set of nonzero real numbers; $(a, b) * (c, d) = (ac, bc + d)$.

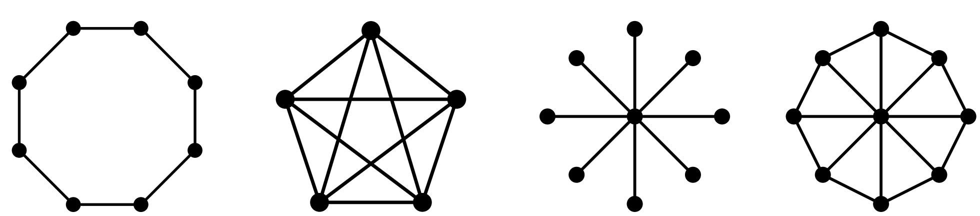
(D2) *Graph automorphisms.* An *automorphism* of a graph H is a permutation σ of the vertices of H such that $\sigma(a)$ and $\sigma(b)$ are connected whenever a and b are connected.

For each of the following graphs H, find $\mathbb{A}(H)$, the group of all automorphisms of H. For each, identify which "key" group from Tuesday's lecture $\mathbb{A}(H)$ is isomorphic to.

For this problem **only**, "handwavy proof by discussion" is permissible.

(a) The cycle graph $G = C_n$ for $n \geq 3$.
(b) The complete graph $G = K_n$ for $n \geq 2$.
(c) The star graph $G = T_n$ for $n \geq 3$.
(d) The wheel graph $G = W_n$ for $n \geq 3$.

Use the following graphs as a guide for this problem.



Cycle graph C_8 Complete graph K_5 Star graph T_8 Wheel graph W_8

Hint: begin by labeling the vertices $1, 2, \ldots$, and record each automorphism as a permutation of the vertex labels.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Consider the dihedral group $D_3 = \{e, r, r^2, f_1, f_2, f_3\}$, which is the symmetry group of the triangle with vertices labeled $1, 2, 3$ when read clockwise, where r denotes a $120°$ clockwise rotation and each f_i denotes the flip leaving vertex i fixed.

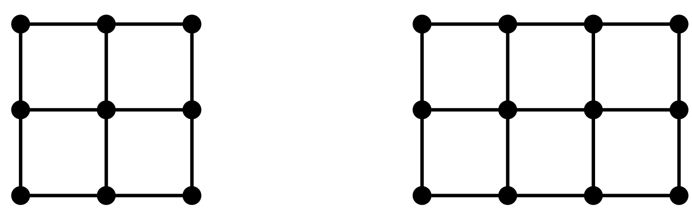
(a) Write each element of D_3 in permutation notation, e.g.

$$f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

(b) Write each element of S_3 in the same permutation notation as part (a). What does this tell you about D_3 and S_3?

(H2) Determine the automorphism group of each of the following graphs (you may either write the elements as permutations, or identify the group as one of the "key" groups from class).

As with problem (D2), for this problem **only**, "handwavy proof by discussion" is allowed.



(H3) Each of the following sets of 2×2 real matrices does **not** form a group under matrix multiplication. For each, locate an axiom that is violated, and give a specific example demonstrating this is the case.

(a) $M = M(\mathbb{R})$ (that is, the set of **all** 2×2 matrices with real entries).

(b) $M = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \text{ with } a \neq 0 \text{ and } b \neq 0 \right\}.$

(H4) Determine whether each of the following sets G form a group under the given operation $*$. If yes, prove they form a group. If no, give a specific example demonstrating that one of the axioms is violated.

(a) G is the set of nonzero real numbers; $a * b = |a| \cdot b$.

(b) $G = \mathbb{R}$; $a * b = a + b + 3$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Identify a graph H whose automorphism group $\mathbb{A}(H)$ is isomorphic to $(\mathbb{Z}_5, +)$.

Hint: to achieve this, the graph should have *exactly* 5 automorphisms, and each non-identity automorphism should yield the identity when applied 5 times.