Spring 2019, Math 596: Problem Set 1
Due: Tuesday, February 5th, 2019
Numerical Semigroups

Discussion problems. The problems below should be worked on in class.

(D1) *Warmup.* In what follows, let $S = \langle 5, 7 \rangle$. Do the following as a group.

(a) Find $\mathsf{m}(S)$ and $\mathsf{e}(S)$.

(b) Write down all of the gaps of S.

(c) Find $\mathsf{F}(S)$ and $\mathsf{g}(S)$.

(d) Find $\mathrm{Ap}(S; 5)$, $\mathrm{Ap}(S; 7)$, and $\mathrm{Ap}(S; 12)$.

(e) Using $\mathrm{Ap}(S; 5)$, determine whether $22 \in S$.

(f) Find $\mathsf{Z}(25)$ and $\mathsf{Z}(5)$.

(g) Find the smallest $n \in S$ with $|\mathsf{Z}(n)| \geq 2$.

(D2) *Numerical semigroups with 2 generators.* Let $S = \langle a, b \rangle$ with $1 < a < b$ and $\gcd(a, b) = 1$. Formlate a conjecture for all parts below, then prove your claims.

(a) Find a formula for $\mathsf{F}(S)$ in terms of a and b.

(b) Characterize the elements of $\mathrm{Ap}(S; a)$.

(D3) *Factorizations of Apéry set elements.* Fix a numerical semigroup S.

(a) For $S = \langle 3, 7 \rangle$, find $\mathsf{Z}(n)$ for every $n \leq 15$. Do the same for $S = \langle 5, 7, 8 \rangle$ and $n \leq 20$. In each, identify which elements lie in $\mathrm{Ap}(S)$.

(b) In the above examples, what distinguishes the factorizations of elements of $\mathrm{Ap}(S)$ from those of the rest of the elements of S?

(c) Develop a criterion for whether a given element $n \in S$ lies in $\mathrm{Ap}(S)$ based on $\mathsf{Z}(n)$.

Homework problems. You are required to submit all of the problems below.

(H1) Let $S = \langle 5, 9, 13 \rangle$. Find each of the following.

(a) $\mathsf{F}(S)$

(b) $\mathsf{g}(S)$

(c) $\mathrm{Ap}(S; 5)$

(d) $\mathsf{Z}_S(52)$

(H2) The goal of this problem is to prove that every numerical semigroup has a unique generating set that is minimal with respect to containment. Fix a numerical semigroup S, define $S^* = S \setminus \{0\}$, and let $A = S^* \setminus (S^* + S^*)$, where $S^* + S^* = \{a + b : a, b \in S^*\}$.

(a) Prove that A generates S.

(b) Prove that every generating set for S has A as a subset.

(c) Prove that A is finite.

(d) Conclude A is the unique generating set for S that is minimal under containment. Hint: why do we need part (c)?

(H3) Fix a numerical semigroup S, and let $m = \mathsf{m}(S)$ and $\mathrm{Ap}(S; m) = \{0, a_1, \ldots, a_{m-1}\}$, where $a_i \equiv i \bmod m$ for each i. Find a formula for $\mathsf{g}(S)$ in terms of $a_1, \ldots, a_{m-1}$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Fix $d \geq 1$ and $a \geq 3$ with $\gcd(a, d) = 1$, and let $S = \langle a, a + d, a + 2d \rangle$. Find a formula for $\mathsf{F}(S)$ in terms of a and d.