Spring 2019, Math 596: Problem Set 2
Due: Tuesday, February 12th, 2019
More Numerical Semigroups

Discussion problems. The problems below should be worked on in class.

(D1) *Warmup.* Let $S = \langle 9, 13, 14, 16 \rangle$.

(a) Find $\mathrm{Ap}(S; 9)$.

(b) Find $\mathsf{PF}(S)$ and $\mathsf{t}(S)$.

(c) Find an irreducible numerical semigroup T containing S with $\mathsf{F}(T) = \mathsf{F}(S)$.

(d) Find all numerical semigroups with Frobenius number 6, organized by containment.

(e) Do the same with Frobenius number 7.

(D2) *Pseudo-Frobenius numbers.* Fix a numerical semigroup S.

(a) Let $T = S \cup \{\mathsf{F}(S)\}$. Determine all possible values of $\mathsf{e}(T)$ (in terms of $\mathsf{e}(S)$).

(b) Suppose $S = \langle a, b \rangle$ with $\gcd(a, b) = 1$. Determine when S is symmetric.

(c) Characterize the set of pseudo-Frobenius numbers of $S = \langle m, m+1, m+2 \rangle$.

Homework problems. You are required to submit all of the problems below. For those with choices, you may submit additional parts if you wish, but please indicate what you want graded. Although I am happy to provide written feedback on all submitted work, no extra credit will be awarded for completing additional problems.

(H1) Let $S = \langle 10, 13, 14, 16 \rangle$. Find $\mathsf{t}(S)$, and determine whether S is symmetric.

(H2) Find all numerical semigroups with genus at most 5 (there should be 12 of genus 5, all in the bottom row). Arrange them as we did at the end of Tuesday's class. Color/bold each edge corresponding to removing the *largest* minimal generator of the previous numerical semigroup. Does this define a rule that matches each numerical semigroup with genus n to a distinct numerical semigroup with genus $n + 1$?

(H3) Complete *at least one* of the following.

- (a) Prove that any numerical semigroup S can be expressed as the intersection of finitely many *irreducible* numerical semigroups (that is, numerical semigroups that are either symmetric or pseudo-symmetric).

  Hint: is every numerical semigroup contained in an irreducible numerical semigroup?
- (b) Prove the intersection of finitely many numerical semigroups is a numerical semigroup. Is the same true for the intersection of infinitely many numerical semigroups?

(H4) Complete *at least one* of the following.

- (a) Characterize all pseudosymmetric numerical semigroups with multiplicity 4.
- (b) Determine whether every numerical semigroup S with $\mathsf{F}(S)$ odd can be expressed as an intersection of symmetric numerical semigroups also with Frobenius number $\mathsf{F}(S)$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Suppose S has embedding dimension 3. Determine how large $\mathsf{t}(S)$ can be.

(C2) Suppose S has embedding dimension 4. Determine how large $\mathsf{t}(S)$ can be.