Spring 2019, Math 596: Problem Set 5
Due: Tuesday, March 5th, 2019
Polytopes

Discussion problems. The problems below should be worked on in class.

(D1) *Polar duals.* The goal of this problem is to practice constructing dual polytopes.

(a) Consider the polytope

$$P = \operatorname{conv}\{(-1,0),(0,-1),(1,-1),(1,1),(-1,1)\}.$$

(i) Find a matrix A so the system of inequalities $Ax \le \vec{1}$ completely describe $x \in P$ (this is the *H-description* of P).

(ii) Find the V-description of P^Δ (the *polar dual* of P).

(iii) Find the H-descriptin of P^Δ, use it to find the V-description of P. Does this match with the original definition of P?

(iv) Write the face lattice of P and P^Δ side by side.

(b) Consider the (slightly modified) simplex

$$S = \operatorname{conv}\{(-1,-1,-1),(1,0,0),(0,1,0),(0,0,1)\} \subset \mathbb{R}^3.$$

Find S^Δ, and use it to show S is *self-dual* (i.e., S has the same face lattice as S^Δ).

(c) Show that a pyramid with square base is self-dual.

(D2) *Proving things about polytopes.* The goal of this problem is to get a feeling for how to write rigorous proofs involving polytopes.

(a) Draw the cubes $C_2 \subset \mathbb{R}^2$ and $C_3 \subset \mathbb{R}^3$. Label the vertices in each drawing.

(b) Formulate a conjecture on when two vertices v_1 and v_2 of the d-dimensional cube C_d are connected by an edge.

The goal of the remainder of this problem is to prove your conjecture for C_3.

(c) For each edge e connecting vertices v_1 and v_2 of C_3, find an equation of a hyperplane H (which should have the form $a_1x_1 + a_2x_2 + a_3x_3 = b$ for some $a_1, a_2, a_3, b \in \mathbb{R}$) so that (i) the only vertices H contains are v_1 and v_2, and (ii) the remaining vertices of C_3 lie on the same side of H. This ensures H is "just touching the polytope" at e.

Hint: be systematic, and use symmetry to your advantage!

(d) For two points $x, y \in \mathbb{R}^d$, let

$$\overline{xy} = \{tx + (1-t)x' : 0 \le t \le 1\}$$

denote the line segment connecting x and y. For each pair of vertices v_1 and v_2 *not* connected by an edge, locate two points w_1 and w_2 in C_3 *not* in $\overline{v_1v_2}$ for which $\overline{w_1w_2} \cap \overline{v_1v_2}$ is nonempty.

Hint: it is possible to choose each w_1 and w_2 to also be vertices of C_3 (this is special to the cube and is *not* true in general).

(e) Briefly explain why the previous part prove $\overline{v_1v_2}$ is not an edge.

(f) If you are feeling adventurous, generalize your constructions above to characterize (with proof) the edges of C_d for $d \ge 3$.

Homework problems. You are required to submit all of the problems below. For those with choices, you may submit additional parts if you wish, but please indicate what you want graded. Although I am happy to provide written feedback on all submitted work, no extra credit will be awarded for completing additional problems.

(H1) Consider the following polytope.

$$P = \operatorname{conv}\{(1,0,0),\ (0,1,0),\ (-1,-1,0),\ (0,0,1),\ (0,0,-1)\}$$

(a) Find a matrix A so that the system of inequalities $Ax \le \vec{1}$ completely describe $x \in P$.

(b) Write down the V-description and H-description of P^Δ.

(c) Draw (as best you can) P and P^Δ.

(d) Draw the face lattice of P and the face lattice of P^Δ.

Note: you are *not* required to prove your descriptions and face lattices are correct.

(H2) Complete *at least one* of the following.

(a) Prove (using hyperplanes) that any two vertices of a d-simplex share an edge.

(b) Characterize (with proof!) all possible f-vectors of 2-dimensional polytopes.

(c) Locate a family of 3-dimensional polytopes that demonstrates #facets - #vertices can be arbitrarily large (a few drawing and/or a brief informal argument is sufficient). Can the same be said for #vertices - #facets?

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) The *permutohedron* is the polytope P_n whose vertices consist of all possible orderings of the coordinates of $(1, 2, \ldots, n)$. For example,

$$P_3 = \{(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)\}.$$

Prove that $\dim P_n = n - 1$.

Hint: to prove $\dim P_n \le n-1$, find a hyperplanes that contain P_n. To prove $\dim P_n \ge n-1$, locate $n-1$ linearly independent vectors of the form $v_2 - v_1$ for vertices v_1, v_2 of P_n.

(C2) Classify the edges of the permutohedron P_n from Problem (C1).