Spring 2019, Math 596: Problem Set 6
Due: Tuesday, March 12th, 2019
Ehrhart Theory

Discussion problems. The problems below should be worked on in class.

(D1) *Computing Ehrhart functions.* Find the Ehrhart function and volume of each polytope.

(a) The 2-dimensional simplex $S_2 = \text{conv}\{(0,0),\ (1,0),\ (0,1)\} \subset \mathbb{R}^2$.

(b) The lattice polygon $P = \text{conv}\{(0,0),\ (1,0),\ (0,1),\ (2,2)\} \subset \mathbb{R}^2$.

(c) The rational polygon $P = \text{conv}\{(1,0),\ (1,\frac{1}{2}),\ (\frac{1}{2},1)\} \subset \mathbb{R}^2$.

(d) The rational polygon $P = \text{conv}\{(0,0),\ (1,\frac{1}{2}),\ (2,0)\} \subset \mathbb{R}^2$. Is your answer a polynomial or a quasipolynomial?

(e) The lattice polytope $P = \text{conv}\{(0,0,0),\ (1,0,0),\ (0,1,0),\ (0,0,1)\} \subset \mathbb{R}^3$.

(D2) *Pick's Theorem.* The goal for this problem is to prove the following.

Theorem (Pick's Theorem)**.** *For any lattice polygon P with I interior lattice points, B boundary lattice points, and area A, we have*

$$A = I + \tfrac{1}{2}B - 1.$$

(a) Verify Pick's theorem for the following polygons.

$$P_1 = \text{conv}\{(0,0),\ (1,0),\ (0,1),\ (2,2)\}$$
$$P_2 = \text{conv}\{(0,0),\ (1,0),\ (1,2),\ (2,2),\ (0,3)\}$$

(b) Suppose $P \subset \mathbb{R}^2$ is a lattice polygon. Use Ehrhart's theorem to write $L_P(t)$ and $L_{P^\circ}(t)$ in terms of B, I, and A (and of course t).

(c) Conclude that Pick's theorem holds.

(d) Using Pick's theorem, find $L_P(t)$ and $L_{P^\circ}(t)$ **without** dilating.

$$P = \text{conv}\{(-1,-1),\ (1,-1),\ (-2,0),\ (2,0),\ (0,2)\}$$

(e) Does Pick's theorem hold for rational polygons?

Homework problems. You are required to submit all of the problems below. For those with choices, you may submit additional parts if you wish, but please indicate what you want graded. Although I am happy to provide written feedback on all submitted work, no extra credit will be awarded for completing additional problems.

(H1) Find the Ehrhart polynomial of

$$P = \text{conv}\{(0,0,0),\ (0,0,1),\ (0,1,0),\ (0,1,1),\ (1,0,0),\ (1,1,0)\},$$

which is a cube with 2 adjacent vertices removed.

(H2) Complete *at least one* of the following.

(a) Fix $a, b \in \mathbb{Z}_{\geq 1}$, and consider the rhombus

$$P = \{(x, y) : a|x| + b|y| \leq ab\}.$$

Find $L_P(t)$ in terms of a, b, and t.

(b) Fix three nonzero vectors $a, b, c \in \mathbb{Z}^3_{\geq 0}$. Use Ehrhart's theorem to prove the following fact from Calculus III: the volume of the parallelopiped formed by a, b, and c is given by $a \cdot (b \times c)$ (these are dot and cross products, respectively).

(c) Read, digest, and transcribe the "triangluation" proof of Pick's theorem given in Section 2.6 of *Computing the Continuous Discretely* by Beck and Robins (this requires filling in some gaps).

(H3) Complete *at least one* of the following.

(a) Find a formula for the volume of the d-dimensional simplex $S_d \subset \mathbb{R}^d$ whose vertices consist of the origin and the standard basis vectors.

(b) Fix two polytopes $P \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$, and define

$$P \times Q = \{(p_1, \ldots, p_n, q_1, \ldots, q_m) : (p_1, \ldots, p_n) \in P, (q_1, \ldots, q_m) \in Q\} \subset \mathbb{R}^{n+m}$$

(you may assume $P \times Q$ is a polytope). Express $L_{P \times Q}(t)$ in terms of $L_P(t)$ and $L_Q(t)$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Let $P \subset \mathbb{R}^d$ denote the d-dimensional cross polytope (octohedron). Find $L_P(t)$.