Spring 2019, Math 596: Problem Set 7
Due: Tuesday, March 19th, 2019
Polytopes and Numerical Semigroups

Discussion problems. The problems below should be worked on in class.

(D1) *Growth rate of factorization counts.*

(a) Suppose $a = (a_1, \ldots, a_k)$, $b = (b_1, \ldots, b_k) \in \mathbb{Z}^k$. Find a formula for the number of integer points on the line segment $\overline{ab}$ in terms of the coordinates of a and b.

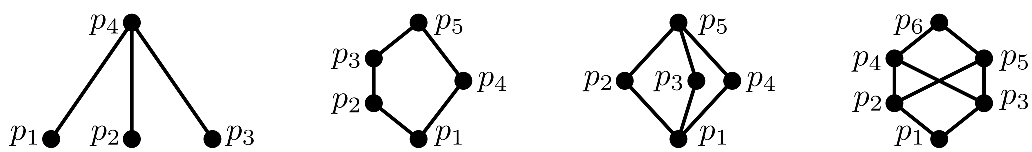
(b) Let $S = \langle 3, 5 \rangle$. Ehrhart's theorem implies $|\mathsf{Z}(n)|$ is quasilinear in n with period 15. Find a formula for the (constant) leading coefficient. Hint: use part (a) to find a formula for $\mathsf{Z}(15n)$ for $n \geq 0$. Why is this sufficient?

(c) Suppose $S = \langle n_1, n_2 \rangle$. Find a formula for the leading coefficient of $|Z(n)|$.

(D2) *The faces of order polytopes.* Given a poset $\Pi = \{p_1, \ldots, p_n\}$, recall that the *order polytope* of Π, denoted $\mathcal{O}(\Pi)$, has H-description

$$\mathcal{O}(\Pi) = \{(a_1, \ldots, a_n) \in \mathbb{R}^n : 0 \leq a_i \leq 1 \text{ for } i = 1, \ldots, n \text{ and } a_i \leq a_j \text{ whenever } p_i \preceq p_j\}.$$

In each part of this problem, the following posets may be helpful examples to consider.



(a) Determine the H-description of $\mathcal{O}(\Pi)$, that is, the *irredundant* list of facet inequalities. Hint: it suffices to eliminate any inequalities that are implied by others. Start by writing out *all* inequalities for each of the above posets, and manually determining which are redundant. Your answer will involve the Hasse diagram of Π.

(b) Complete the following proof that every vertex of $\mathcal{O}(\Pi)$ lies in $\{0, 1\}^n$ (that is, each coordinate must be either 0 or 1).

Proof. Suppose $a \in \mathcal{O}(\Pi)$, and let $S = \{i : 0 < a_i < 1\}$. We will show that if $S \neq \emptyset$, then a is contained in a line segment that is contained in $\mathcal{O}(\Pi)$. Letting

$$m = \min\{a_i : i \in S\} \qquad \text{and} \qquad M = \max\{a_i : i \in S\},$$

we claim

$$a' = a - m \sum_{i \in S} e_i \qquad \text{and} \qquad a'' = a + (1 - M) \sum_{i \in S} e_i$$

both lie in $\mathcal{O}(\Pi)$. For each i we have $a'_i = a_i$ if $i \notin S$, and

$$0 \leq a_i - m \leq a'_i \leq a_i \leq 1$$

if $i \in S$. Moreover, for any i, j with $p_i \preceq p_j$, we have either

$$a'_i = a_i - m \leq 1 = a_j, \qquad a_i = 0 \leq a_j - m = a'_j, \qquad \text{or} \qquad a'_i = a_i - m \leq a_j - m = a'_j$$

based on whether just $i \in S$, just $j \in S$, or both $i, j \in S$. As such, $a' \in \mathcal{O}(\Pi)$. Similarly, _______, so $a'' \in \mathcal{O}(\Pi)$. Since a' and a'' both lie in $\mathcal{O}(\Pi)$, the line segment between them (which contains a) also lives in $\mathcal{O}(\Pi)$. This completes the proof. $\square$

(c) The above result implies that the vertices of $\mathcal{O}(\Pi)$ are precisely the 01-vectors that satisfy the inequalties defining $\mathcal{O}(\Pi)$. In particular, we can associate each vertex v of $\mathcal{O}(\Pi)$ to a subset of Π, e.g., $v = (1, 0, 0, 1, 0, 1)$ corresponds to the set $\{p_1, p_4, p_6\}$. Characterize which subsets of Π coorespond to vertices of $\mathcal{O}(\Pi)$.

Homework problems. You are required to submit all of the problems below. For those with choices, you may submit additional parts if you wish, but please indicate what you want graded. Although I am happy to provide written feedback on all submitted work, no extra credit will be awarded for completing additional problems.

(H1) (a) Determine a complete list of 3-element posets (there are 5 distinct Hasse diagrams, and only one poset for each is required).

(b) Determine the order polynomial $\Omega_\Pi(t)$ of each poset Π obtained in part (a). You may use the fact that $\Omega_\Pi(t) = L_{\mathcal{O}(\Pi)}(t-1)$ as in (H2) below.

(c) Determine the H-description and V-description of the order polytope of each poset obtained in part (a).

(d) (Optional) Draw the order polytope of each poset obtained in part (a), or make a computer do it for you (see the announcement at the bottom of this page).

(H2) Explain why, for any finite poset Π, we have $\Omega_\Pi(t) = L_{\mathcal{O}(\Pi)}(t-1)$ and $\Omega_\Pi^\circ = L_{\mathcal{O}(\Pi)^\circ}(t-1)$.

(H3) Complete *at least one* of the following.

(a) Let $S = \langle n_1, n_2, n_3 \rangle$. By Ehrhart's theorem, $|\mathsf{Z}(n)|$ is a quasipolynomial of degree 2 with period $\operatorname{lcm}(n_1, n_2, n_3)$ whose leading coefficient is constant. Conjecture a formula for the leading coefficient in terms of n_1, n_2, and n_3.

Hint: this is possible to do by hand, but requires some care. Start with **small** examples (e.g., $S = \langle 3, 4, 5 \rangle$), and only find $|\mathsf{Z}(n)|$ when n is an integer multiple of $\operatorname{lcm}(n_1, n_2, n_3)$ (including $n = 0$). Alternatively, you can use `Sage` to count the factorizations for you.

(b) Suppose $\Pi = \{p_1, \ldots, p_d\}$ is a finite poset. The vertices of $\mathcal{O}(\Pi)$ are precisely the 01-vectors that satisfy the inequalties defining $\mathcal{O}(\Pi)$. In particular, we can associate each vertex v of $\mathcal{O}(\Pi)$ to a subset of Π, e.g., a vertex $v = (1, 0, 0, 1, 0, 1)$ corresponds to the set $\{p_1, p_4, p_6\}$. Characterize which subsets of Π coorespond to vertices of $\mathcal{O}(\Pi)$ in terms of the Hasse diagram of Π (this is essentially (D2)(c)).

(c) Prove that for any finite poset $\Pi = \{p_1, \ldots, p_n\}$, we have $\dim \mathcal{O}(\Pi) = n$.

Hint: find n pairs of vertices (v, v') whose differences $v - v'$ are linearly independent.

(d) Read, digest, and transcribe the direct (i.e., polytope-free) proof in Proposition 1.3.1 of *Combinatorial Reciprocity Theorems* by Beck and Sanyal, which states that the number of strictly order preserving functions from any poset P to $\{1, \ldots, t\}$ is a polynomial in t (this requires filling in some gaps).

Note: a proof using Ehrhart's theorem and order polytopes is given later in this book, and would make a great reading project!

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Given a poset P, characterize the edges of the order polytope $\mathcal{O}(P)$.

Announcement: for those who are tired of drawing polytopes by hand, there are a couple free (web-based) programs you can use instead. The first was developed during the last few weeks by Nils Olsson (who has generously offered to share his creation).

`https://nilsso.github.io/pages/math/semi-comb/polytope.html`

The second is `Geogebra`, a more general geometry application.

`https://www.geogebra.org/3d?lang=en`

`Geogebra` is a great tool for rendering and visualizing 2D and 3D geometric objects, including polytopes. The current version seems to require using the "polygon" command for each facet, but a full "polytope" function appears to be on the horizon.