Spring 2019, Math 596: Problem Set 8
Due: Tuesday, March 26th, 2019
Enumerating Numerical Semigroups Using Polyhedra

Discussion problems. The problems below should be worked on in class.

(D1) *The Kunz polyhedron and Wilf's conjecture.* Recall that for each $m \geq 2$, the Kunz polyhedron $P_m \subset \mathbb{R}^{m-1}$ has defining inequalities

$$\begin{array}{rll} x_i \geq 1 & \text{for} & 1 \leq i \leq m-1, \\ x_i + x_j \geq x_{i+j} & \text{for} & 1 \leq i \leq j \leq m-1 \text{ with } i+j < m, \text{ and} \\ x_i + x_j + 1 \geq x_{i+j} & \text{for} & 1 \leq i \leq j \leq m-1 \text{ with } i+j > m. \end{array}$$

(a) Write the H-description of P_3, and draw it.

(b) Locate a numerical semigroup in each face, and write down their generating sets. If possible, choose one in the relative interior of each face.

(c) A generalization of Ehrhart's theorem implies that the function

$$L(t) = \left|\mathbb{Z}_{m-1} \cap \{(x_1, \ldots, x_{m-1}) \in P_m : x_1 + \cdots + x_{m-1} = t\}\right|$$

equals a quasipolynomial once t is sufficiently large. Compute $L(t)$ for P_3, and in particular locate the smallest value of t after which $L(t)$ is quasipolynomial.

(d) Is $L(t)$ in the previous part non-decreasing for all $t \geq 1$? How does this relate to the "sad" genus conjecture from several weeks ago?

(e) Locate at least 3 numerical semigroups in each *unbounded* face of P_3.

(f) Compute the Apéry set of each numerical semigroup, and draw the Hasse diagram (remember this has $a_i \preceq a_j$ whenever $a_j - a_i \in S$). Do these match as we saw in class?

(g) Compute the genus and Frobenius number of each numerical semigroup thus far.

(h) Find a formula for the genus and Frobenius number of the numerical semigroup in terms of its Kunz coordinates $(x_1, \ldots, x_{m-1})$.

(i) Recall that Wilf's conjecture claims every numerical semigroup S satisfies

$$\mathsf{F}(S) + 1 \leq \mathsf{e}(S)(\mathsf{F}(S) - \mathsf{g}(S) + 1).$$

Verify Wilf's conjecture for every numerical semigroup lying in a bounded face of P_3.

(j) For each unbounded 1-dimensional face F, do the following.

(i) Write $\mathsf{g}(S)$, $\mathsf{F}(S)$, and $\mathsf{e}(S)$ in terms of the Kunz coordinates $(x_1, x_2) \in F$.

(ii) Rewrite Wilf's inequality (for semigroups within F) in terms of x_1 and x_2.

(iii) Plot Wilf's inequality on the same axes as P_3. Locate any integer points in the relative interior of F that satisfy Wilf's inequality.

(k) Repeat the above steps for the relative interior of P_3. Note that you will need to consider 2 cases, based on which coordinate corresponds to the Frobenius number.

(l) Conclude that Wilf's conjecture holds for every numerical semigroup of multiplicity 3.

Homework problems. You are required to submit all of the problems below. For those with choices, you may submit additional parts if you wish, but please indicate what you want graded. Although I am happy to provide written feedback on all submitted work, no extra credit will be awarded for completing additional problems.

(H1) Complete *at least one* of the following.

(a) Suppose $x_1, \ldots, x_{m-1} \in \mathbb{Z}_{\geq 1}$. Prove that

$$\{0, mx_1 + 1, \ldots, mx_{m-1} + (m-1)\}$$

is the Apery set of a numerical semigroup if and only if $(x_1, \ldots, x_{m-1})$ satisfies the defining inequalities of the Kunz polyhedron P_m.

(b) Complete any part of a past homework problem that you did not already complete.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Complete parts (e) through (l) from discussion, wherein Wilf's conjecture (stated in part (i)) is verified for all multiplicity 3 numerical semigroups using the Kunz polyhedron P_3.

(C2) Use the ideas from the second half of discussion to verify that Wilf's conjecture holds for every numerical semigroup of multiplicity 4.

Hint: you can use Sage to obtain the face lattice of P_4 via the `Polyhedron` object.