## Spring 2019, Math 596: Problem Set 10 <br> Due: Tuesday, April 16th, 2019 <br> Introduction to Generating Functions

Discussion problems. The problems below should be worked on in class.
(D1) Recurrence relations. Solve each of the following recurrences with generating functions.
(a) $a_{0}=1, a_{n}=2 a_{n-1}$.
(b) $a_{0}=3, a_{1}=1, a_{n}=2 a_{n-1}+3 a_{n-2}$.
(D2) Power series of polynomial functions. Recall that in lecture, we saw

$$
1+z+z^{2}+\cdots=\sum_{n \geq 0} z^{n}=\frac{1}{1-z}
$$

The following identity can be obtained by "differentiating both sides" using the standard differentiation rules from calculus.

$$
1+2 z+3 z^{2}+\cdots=\sum_{n \geq 0}(n+1) z^{n}=\frac{1}{(1-z)^{2}}
$$

(a) Verify this identity by multiplying the series $1 /(1-z)$ by itself.
(b) Find an expression for $\sum_{n \geq 0} n z^{n}$ as a rational function of $z$.
(c) Use "formal differentiation" to express $\sum_{n \geq 0} n^{2} z^{n}$ as a rational function of $z$.
(d) Use "formal differentiation" to express $\sum_{n \geq 0} n^{3} z^{n}$ as a rational function of $z$.
(D3) Formal power series derivatives. Given $F(z)=f_{0}+f_{1} z+f_{2} z^{2}+\cdots$, define

$$
F^{\prime}(z)=\frac{d}{d z} F(z)=f_{1}+2 f_{2} z+3 f_{3} z^{2}+\cdots
$$

as the "formal derivative" of $F(z)$. In general, it turns out this formal derivative operation satisfies all of the standard calculus rules for derivatives. The goal of this problem is to prove some of these identities.

Fix formal power series $F(z)$ and $G(z)$.
(a) Prove that $\frac{d}{d z}[F(z) G(z)]=F^{\prime}(z) G(z)+F(z) G^{\prime}(z)$.
(b) Prove that if $G$ has no constant term, then $\frac{d}{d z}[F(G(z))]=F^{\prime}(G(z)) G^{\prime}(z)$.
(c) Prove that if $G(z)$ has nonzero constant term, then

$$
\frac{d}{d z}\left[\frac{F(z)}{G(z)}\right]=\frac{F^{\prime}(z) G(z)-F(z) G^{\prime}(z)}{(G(z))^{2}} .
$$

Hint: using parts (a) and (b) can save a LOT of algebra.
(d) Where in parts (b) and (c) did you use the assumption(s) on $G(z)$ ?

Homework problems. You are required to submit all of the problems below. For those with choices, you may submit additional parts if you wish, but please indicate what you want graded. Although I am happy to provide written feedback on all submitted work, no extra credit will be awarded for completing additional problems.
(H1) Complete at least one of the following.
(a) Use generating functions to find an explicit formula for $L_{n}$ if $L_{0}=2, L_{1}=1$, and $L_{n}=L_{n-1}+L_{n-2}$. These are called the Lucas numbers.
(b) Let

$$
A(z)=\sum_{n \geq 0} \frac{(-1)^{n}}{(2 n+1)!} z^{2 n+1}=z-\frac{1}{6} z^{3}+\cdots \quad \text { and } \quad B(z)=\sum_{n \geq 0} \frac{(-1)^{n}}{(2 n)!} z^{2 n}=1-\frac{1}{2} z^{2}+\cdots
$$

Show that $(A(z))^{2}+(B(z))^{2}=1$. Thinking back to Calculus 2 , what familiar identity does this equality encode?
(c) Complete any one of the proofs from (D3) that you did not complete in class.
(H2) Make partial progress on at least one of the following.
(a) As is hinted by Problem (D2), for each $d \geq 1$, we have

$$
\sum_{n \geq 0} n^{d} z^{n}=\frac{Q(z)}{(1-z)^{d+1}}
$$

for some polynomial $Q(z)$. This can be proven by induction on $d$ (you do not need to prove this). Find a general formula for $Q(1)$ in terms of $d$.
(b) Fix a formal power series $F(z)=\sum_{n=0}^{\infty} f_{n} z^{n}$ with coefficients in $\mathbb{Q}$. Characterize when $F(z)$ has a multiplicative inverse (that is, when there exists another formal power series $G(z)$ such that $F(z) G(z)=1)$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.
(C1) Fix a function $f: \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}$. Prove that we can express the generating function

$$
F(z)=\sum_{n=0}^{\infty} f(n) z^{n}=\frac{Q(z)}{(1-z)^{d+1}}
$$

for some polynomial $Q(z)$ of degree at most $d$ if and only if $f(n)=a_{d} n^{d}+\cdots+a_{1} n+a_{0}$ is a polynomial of degree exactly $d$. Hint: start with $Q(z)=1$.

