Spring 2019, Math 596: Problem Set 11
Due: Tuesday, April 23th, 2019
Rational Generating Functions

Discussion problems. The problems below should be worked on in class.

(D1) *Factorizations in numerical semigroups.* Recall that in lecture, we saw

$$\sum_{n=0}^{\infty} a(n)z^n = \left(\frac{1}{1-z^{n_1}}\right)\cdots\left(\frac{1}{1-z^{n_k}}\right),$$

where $a(n)$ denotes the number of factorizations of n in the numerical semigroup $\langle n_1, \ldots, n_k\rangle$. For each function $a(n)$ described below, express the generating function $\sum_{n\geq 0} a(n)z^n$ using rational functions of z (in the flavor of the one above). Write out the first several nonzero terms of each and verify they match the desired integers.

(a) $a(n)$ is the number of ways to write n as a sum of 6's, 9's, and 20's.
(b) $a(n)$ is the number of ways to write n as a sum of 6's, 9's, and 20's using an odd number of 6's.
(c) $a(n)$ is the number of ways to write n as a sum of 6's, 9's, and 20's using exactly one 9.
(d) $a(n)$ is the number of ways to write n as a sum of 6's, 9's, and 20's using the same number of 9's and 20's.
(e) $a(n)$ is the number of ways to write n as a sum of 1's, 2's, ..., m's (for fixed $m \geq 1$).
(f) $a(n)$ is the number of ways to write n as a sum of positive integers. Hint: your answer may involve an infinite product! We will look more at this next week.

(D2) *Computing Ehrhart series.* Recall from lecture that if $P \subset \mathbb{R}^d$ is a polytope, then

$$\mathrm{Ehr}_P(z) = \sum_{n\geq 0} L_P(n)z^n$$

is the *Ehrhart series* of P.

(a) Find the Ehrhart polynomial $L_P(t)$ of the pentagon

$$P = \mathrm{conv}\{(-1,-1),(1,-1),(-1,0),(1,0),(0,1)\}.$$

Use this to find the Ehrhart series of P.

(b) Fix integers $a < b$, and let $P = [a,b] \subset \mathbb{R}$, that is, the closed interval from a to b (this is a 1-dimensional polytope with vertices a and b). Find the Ehrhart series of P in terms of a and b.

(c) Consider the rational polygon

$$P = \mathrm{conv}\{(0,0),(\tfrac{1}{2},0),(0,\tfrac{1}{2})\}.$$

Find a formula for $L_P(t)$ (remember, this will be a *quasipolynomial*). Use this to show that numerator of the Ehrhart series of P is just $1+z$.

(D3) *Ehrhart series of lattice polygons.* Suppose $P \subset \mathbb{R}^2$ is a lattice polygon with area A, I interior lattice points, and B boundary lattice points.

(a) Use Pick's theorem ($A = I + \frac{1}{2}B - 1$) to find the coefficients of $L_P(t)$ in terms of I, B.
(b) Find the Ehrhart series of P, again in terms of I and B.
(c) Verify both of your formulas match Problem (D1)(a).
(d) Prove that the numerator of $\mathrm{Ehr}_P(z)$ has non-negative integer coordinates.

Homework problems. You are required to submit all of the problems below. For those with choices, you may submit additional parts if you wish, but please indicate what you want graded. Although I am happy to provide written feedback on all submitted work, no extra credit will be awarded for completing additional problems.

(H1) Complete *at least one* of the following.

(a) Let $P = \text{conv}\{(\frac{1}{2}, 0), (0, \frac{1}{2})\}$. Find the Ehrhart series of P.

(b) Find the Ehrhart series of the 2-dimensional permutohedron

$$P_3 = \{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}.$$

Remember: this is a "flat" hexagon living in $\mathbb{R}^3$!

(H2) Make partial progress on *at least one* of the following.

(a) Fix $a, b \in \mathbb{Z}_{\geq 2}$ with $\gcd(a, b) = 1$, and let $S = \langle a, b \rangle$. Let

$$h(n) = \begin{cases} 1 & \text{if } n \in S; \\ 0 & \text{if } n \notin S. \end{cases}$$

Find an expression for $Q(z)$ in terms of a and b, where

$$\sum_{n \geq 0} h(n) z^n = \frac{Q(z)}{1 - z^a}.$$

(b) Use notation from the previous problem, and let

$$\sum_{n \geq 0} h(n) z^n = \frac{R(z)}{(1 - z^a)(1 - z^b)}.$$

Find an expression for $R(z)$ in terms of a and b.

(c) Find a rational expression for the Ehrhart series of the $(d-1)$-dimensional simplex

$$S_d = \text{conv}\{\vec{e}_1, \ldots, \vec{e}_d\} \subset \mathbb{R}^d$$

Note: this is *not* the simplex we have used previously, as the origin is not a vertex!

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Let P be an integral polygon. Prove that

$$\text{Ehr}_P(z) = \frac{Q(z)}{(1 - z)^3}$$

where $Q(z)$ has non-negative integer coefficients, **without** using Pick's theorem.

Hint: use triangulations to reduce to the case when P contains exactly 3 integer points.