Spring 2019, Math 596: Problem Set 12
Due: Tuesday, April 30th, 2019
Rational Generating Functions (Week 2: Deep Cuts)

Discussion problems. The problems below should be worked on in class.

(D1) *Practice with formal power series.* Recall that in lecture, we saw

$$\sum_{n=0}^{\infty} p(n)z^n = \prod_{i=1}^{\infty} \frac{1}{1-z^i},$$

where $p(n)$ denotes the number of partitions of n (that is, the number of ways to write n as a sum of positive integers).

(a) For each function $a(n)$ described below, express the generating function $\sum_{n\geq 0} a(n)z^n$ using rational functions of z (in the flavor of the one above). Write out the first few nonzero terms of each and verify they match the desired integers.

(i) $a(n)$ is the number of ways to write n as a sum of 3, 5, and 7.

(ii) $a(n)$ is the number of ways to write n as a sum of 3, 5, and at most 4 copies of 7.

(iii) $a(n)$ is the number of ways to write n as a sum of positive odd integers.

(iv) $a(n)$ is the number of ways to write n as a sum of positive even integers, where each integer used occurs an odd number of times.

(v) $a(n)$ is the number of ways to write n as a sum of distinct positive integers.

(vi) $a(n)$ is the number of ways to write n as a sum of distinct odd primes.
Note: the terms in this sum are related to the Goldbach conjecture!

(b) Explain why this same technique cannot be used to count the number of ways to express n as a sum of non-negative even integers.

(c) Use algebra to prove that the functions in parts (iii) and (v) are equal for all $n \geq 0$.

(D2) *Ehrhart series of the cross polytope.* Given a polytope $P = \text{conv}\{\vec{v}_1, \ldots, \vec{v}_k\} \subset \mathbb{R}^d$, define

$$\text{Pyr}(P) = \text{conv}\{(\vec{v}_1, 0), (\vec{v}_2, 0), \ldots, (\vec{v}_k, 0), (\vec{0}, 1)\} \subset \mathbb{R}^{d+1} \quad \text{and}$$
$$\text{Bipyr}(P) = \text{conv}\{(\vec{v}_1, 0), (\vec{v}_2, 0), \ldots, (\vec{v}_k, 0), (\vec{0}, 1), (\vec{0}, -1)\} \subset \mathbb{R}^{d+1},$$

the *pyramid* and *bipyramid* over P, respectively (each lives in one dimension more than P).

(a) Let $P = [-1, 1] \subset \mathbb{R}$. Draw P, $\text{Pyr}(P)$ and $\text{Bipyr}(P)$.

(b) Let $P = \text{conv}\{(1, 0), (-1, 0), (0, 1), (0, -1)\}$. Draw P, $\text{Pyr}(P)$ and $\text{Bipyr}(P)$.

(c) Let $A(z) = \sum_{n\geq 0} a_n z^n$ and $B(z) = \sum_{n\geq 0} b_n z^n$, and suppose each $b_n = a_0 + \cdots + a_n$. Find an equation relating $A(z)$ and $B(z)$ that uses no sigma-sums.

(d) Find a formula for $\text{Ehr}_{\text{Pyr}(P)}(z)$ in terms of $\text{Ehr}_P(z)$ (again involving no sigma-sums).

(e) Find a sigma-sum free equation relating the Ehrhart series of P, $\text{Pyr}(P)$, and $\text{Bipyr}(P)$. Use this to obtain a formula for $\text{Ehr}_{\text{Bipyr}(P)}(z)$ in terms of $\text{Ehr}_P(z)$.

(f) Find a formula for the Ehrhart series of the d-dimensional cross polytope

$$P_d = \text{conv}\{\vec{e}_1, -\vec{e}_1, \vec{e}_2, -\vec{e}_2, \ldots, \vec{e}_d, -\vec{e}_d\} \subset \mathbb{R}^d.$$

Homework problems. You are required to submit all of the problems below. For those with choices, you may submit additional parts if you wish, but please indicate what you want graded. Although I am happy to provide written feedback on all submitted work, no extra credit will be awarded for completing additional problems.

(H1) Peer edit the reading paper you were given during discussion.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Fix $d \geq 2$, and let

$$P = \mathrm{conv}\{\vec{e}_i - \vec{e}_j : 1 \leq i, j \leq d \text{ with } i \neq j\} \subset \mathbb{R}^d.$$

Find the Ehrhart series of P.

(C2) The n-th Catalan number C_n, given by

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$

equals the number of binary trees on n vertices (along with **many** other things, a scan will be sent through email with a compendium of examples). Let $C(z) = \sum_{n\geq 0} C_n z^n$ denote the corresponding generating function. Prove that

$$C(z) = 1 + z(C(z))^2$$

using **any** of the following techniques:

(i) algebraically, by substituting the formula for C_n;

(ii) combinatorially, using interpretations of generating function multiplication; or

(iii) inductively, by first proving that C_n satisfies the recurrence $C_0 = 1$ and

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$$

for every $n \geq 0$.

Note: these methods are not completely disjoint, and ideas from each can be combined in numerous ways to obtain a complete proof.