Spring 2019, Math 596: Problem Set 13
Due: Tuesday, May 7th, 2019
Hilbert Series of Numerical Semigroups

Discussion problems. The problems below should be worked on in class.

(D1) *Simplicial complexes.*

(a) Draw each of the following simplicial complexes. Find the dimension and Euler characteristic of each. Do any of these look familiar? Which are topologically equivalent?

(i) $\{123, 124, 134\}$
(ii) $\{123, 124, 134, 234\}$
(iii) $\{12, 13, 14, 15, 16, 23, 34, 45, 56, 26\}$
(iv) $\{123, 134, 145, 125, 236, 346, 456, 256\}$
(v) $\{123, 145, 16, 34, 56, 26\}$

(b) Find a triangulation of the Möbius band (i.e. a 2-dimensional strip of ribbon with a single twist). Do this in two different ways (with a different number of triangles), and verify that the Euler characteristic is the same in each case.

(c) Fix a positive integer n, and define a simplicial complex Δ on the ground set $\{1, 2, \ldots, n\}$ so that $G \in \Delta$ whenever G forms a minimal generating set for a numerical semigroup.

(i) Prove that Δ is a simplicial complex.
(ii) Find the dimension of Δ in terms of n.
(iii) Is Δ *pure* (i.e. are all of its facets the same dimension)? If not, how big are the largest and smallest facets?

(D2) *Hilbert series.*

(a) For each numerical semigroup S below, find all 3 rational function representations of the Hilbert series of S using The Big Theorem. Verify the ∇-complexes from the third from have the correct Euler characteristic.

(i) $S = \langle 3, 5, 7 \rangle$.
(ii) $S = \langle 5, 11, 12, 13 \rangle$.

(b) Suppose $S = \langle a, a + 1, \ldots, 2a - 1 \rangle$. Characterize (in terms of a) the numerator of the Hilbert series of S with each denominator from The Big Theorem.

Homework problems. You are required to submit all of the problems below. For those with choices, you may submit additional parts if you wish, but please indicate what you want graded. Although I am happy to provide written feedback on all submitted work, no extra credit will be awarded for completing additional problems.

(H1) Complete *all* of the following.

(a) Draw the simplicial complex Δ with facets

$$\{123, 234, 345, 456, 567, 678, 178, 128, 15\}.$$

Find the dimension and Euler characteristic of Δ.

(b) Find all 3 rational expressions of $\mathcal{H}(S; z)$ in The Big Theorem for $S = \langle 5, 7, 9\rangle$. Writing

$$\mathcal{H}(S; z) = \frac{Q(z)}{(1 - z^5)(1 - z^7)(1 - z^9)},$$

draw the simplicial complex Δ_n for each term t^n appearing in $Q(z)$.

(H2) Make partial progress on *at least one* of the following.

(a) Fix a numerical semigroup S and an element $n \in S$. Characterize the numerator of the Hilbert series of S when the denominator is $1 - z^n$.

(b) Suppose $S = \langle a, a + 1, a + 2\rangle$. Characterize (in terms of a and k) the numerator of the Hilbert series of S with each denominator from The Big Theorem.

(c) Write a `Sage` function that takes as input a numerical semigroup S, and returns the numerator of the Hilbert series of S (with denominator $1 - z$). Add an optional parameter to your function which allows the user to specify the denominator.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Construct a 2-dimensional simplicial complex that retracts onto both an annulus and a Möbius band.

(C2) Recall that the torus is the space obtained by identifying opposite sides of a square (imagine curling a square piece of paper to form a cylinder, then bending it around to form a hollow donut). Find a triangulation of the torus with the smallest number of triangles.