Spring 2019, Math 596: Project Topics

The goal of each project is to learn about a topic not discussed in class, and/or investigate a related open question. Throughout the semester, the following will be expected.

- Choose a topic. Please speak with me before making your decision, to ensure it is an appropriate level and so that we can narrow down a reasonable set of goals. You should choose a topic (and have it approved) no later than **Friday, March 22nd**.
- Begin reading the agreed-upon background material. Plan to **meet at least twice** with me throughout the rest of the semester, to ensure that you are on track.
- Write (in LaTeX) a paper aimed at introducing your topic to fellow students, containing ample examples and explanations in addition to any theorems and proofs you give. Your writing should convey that you understand the intricacies of any proofs presented. Keep the following deadlines in mind as you proceed.
  - A rough draft of the paper will be due **Thursday, April 25th** (two weeks before the last day of class). This will be peer reviewed by a fellow student in subsequent weeks.
  - The final paper will be due on **Thursday, May 16th** (the day of our final exam).
- Graduate students only: give a 10-15 minute presentation introducing the main ideas of your topic. Presentations will take place during the final exam slot at the semester's end. You should keep in mind your target audience and available time when deciding what and how to present.

  Note: although the presentation is only required for graduate students, everyone is encouraged to present and should strongly consider doing so. If there is sufficient interest, we can spend some time during the last week of classes on presentations as well.
- Your final grade on the project will be determined by the content, quality, and completeness of your final writeup (and, for graduate students, on the quality of the presentation).

Given below are several project ideas. Many of the listed sources contain more material than is necessary for the project, so be sure to meet with me so we can set reasonable project goals. I am also open to projects not listed here, but please run them by me before making a decision. Don't be afraid to ask questions at any point during the project!

Numerical semigroups

(1) *Generalized arithmetical numerical semigroups.* A numerical semigroup with generators of the form $S = \langle a, ah + d, ah + 2d, \ldots, ah + kd \rangle$ (called a *generalized arithmetic sequence*). Such semigroups satisfy a membership criterion (that is, a quick easy test for whether a given integer n satisfies $n \in S$), yielding a concise formula for the Frobenius number of S.

Source: *On the type and minimal presentation of certain numerical semigroups* (M. Omidali, F. Rahmuti).

(2) *Exploring the tree of numerical semigroups.* Efforts to count or estimate the number of numerical semigroups with certain fixed parameters (e.g. fixed genus, or fixed multiplicity and Frobenius number) have provided a wealth of interesting questions. Some methods utilize polyhedral geometry (e.g. the Kunz polyhedron from class) while others organize numerical semigroups systematically by containment. The latter is particularly effective for counting numerical semigroups with a fixed genus, as semigroups at the same "height" have the same genus.

Source: *Fundamental gaps in numerical semigroups* (J. Rosales, P. García-Sánchez, J. García-García, J. Jiménez Madrid).

(3) *Shifted numerical semigroups.* Fix positive integers $r_1, \ldots, r_k$. For each t, consider the numerical semigroup $M_t = \langle t, t + r_1, \ldots, t + r_k \rangle$. The idea is to think of this as a family of numerical semigroups, one for each t, and examine properties like $F(M_t)$ as a function of t. It turns out when t is sufficiently large, $F(M_t)$ is quasiquadratic in t.

Source: *Apéry sets of shifted numerical monoids* (C. O'Neill, R. Pelayo).

(4) *Minimal presentations.* Sometimes referred to as a Markov basis, a *minimal presentation* is a collection of minimal relations between the generators of a semigroup. For example, in the McNugget semigroup $S = \langle 6, 9, 20 \rangle$, one possible minimal presentation is comprised of 2 relations: $((3, 0, 0), (0, 2, 0))$ (i.e. exchanging 3 copies of 6 for 2 copies of 9) and $((1, 6, 0), (0, 0, 3))$ (i.e. exchanging 6 copies of 9 and 1 copy of 6 for 3 copies of 20). Although they require a bit of technical machinery to define formally, minimal presentations are a powerful tool when working with numerical semigroups, both for theoretical results and for explicit computation.

Source: *Numerical semigroups and applications* (J. Rosales, P. García-Sánchez), Chapter 4.

(5) *Elasticity.* The *elasticity* of a semigroup element, defined as the ratio of its maximum factorization length to its minimum factorization length, is a measure of the "spread" of factorization lengths. Elasticity arises most commonly in number theory when studying non-unique factorization, but has been examined in other settings as well, and several surprisingly concise results have been obtained for numerical semigroups.

Source: *Full elasticity in atomic monoids and integral domains* (S. Chapman, M. Holden, T. Moore).

(6) *Dynamic algorithms.* The set of factorizations, as well as several quantities derived from them, can be computed rather quickly using dynamic algorithms (that is, inductive algorithms that save time by storing output for smaller input values). These algorithms are especially efficient for computing length sets, as they do not require computing the full set of factorizations (typically orders of magnitude bigger than the length set).

Source: *On dynamic algorithms for factorization invariants in numerical monoids* (T. Barron, C. O'Neill, B. Pelayo).

Note: for this project, some familiarity with computational complexity will be required.

(7) *Computing the Frobenius number.* One of the oldest questions on numerical semigroups concerns the Frobenius number $F(S)$, that is, the largest integer outside of S. There are several known algorithms for computing $F(S)$ from a list of generators $n_1, \ldots, n_k$ for S, with different computational complexities (although one can prove that no "fast" algorithm is possible in general).

Source: *The Diophantine Frobenius problem* (J. Alfonsín), Chapter 1.

Note: for this project, some familiarity with computational complexity will be required.

(8) *Bounding the Frobenius number.* One of the oldest questions on numerical semigroups concerns the Frobenius number $F(S)$, that is, the largest integer outside of S. In general, no fast algorithm is possible for computing $F(S)$ from a list of generators $n_1, \ldots, n_k$ for S, but there are several bounds, as well as formulas for particularly nice generating sets.

Source: *The Diophantine Frobenius problem* (J. Alfonsín), Chapter 3.

(9) *Puiseux semigroups.* A natural generalization of numerical semigroups, *Puiseux semigroups* are subsemigroups of $(\mathbb{Q}_{\geq 0}, +)$. Unlike numerical semigroups, though, Puiseux semigroups need not have finite generating sets, and moreover need not have minimal generating sets! For instance, $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ generates $S = \mathbb{Q}_{\geq 0}$, but every generator therein is redundant since $\frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}$ for each $n \geq 1$.

Source: *Three families of dense Puiseux monoids* (F. Gotti, M. Gotti, H. Polo).

(10) *Random numerical semigroups.* A recent paper examined numerical semigroups S whose generators are "randomly chosen" via the following process: fix a probability $p \in [0, 1]$ and a positive integer $M \in \mathbb{Z}_{\geq 1}$, and for each integer $n = 1, 2, \ldots, M$, decide with probability p whether or not to include n as a generator for S. For example, if $p = \frac{1}{10}$ and $M = 40$, then on average we expect to choose 4 generators for S, although (i) it is possible to choose as few as 0 generators and as many as 40 (though this is the least likely generating set), and (ii) the generators chosen may not be minimal (e.g. if 6, 9, 20, and 26 are chosen). The key questions of interest have the form "what properties to we expect S to have?" For example, what is the expected number of minimal generators, or the expected Frobenius number? (Here, "expected number" refers to *expected value* from probability.)

It would be interesting to choose a variant of the "random process" defined above, and use computer software to simulate it with a large sample size to estimate the expected number of minimal generators or the expected Frobenius number. For instance, one could allow an additional input m for the multiplicity and only select integers $n = m + 1, \ldots, M$ as additional generators, or allow the probability p to change for different n.

Source: *Randomly generated numerical semigroups (undergraduate thesis)* (Z. Spaulding).
`https://github.com/coneill-math/rns-db-plot`

Note: this project is computational in nature, so some programming will be required.

(11) *Leamer monoids.* Leamer monoids recently arose in the pursuit of a longstanding conjecture from commutative algebra. Each Leamer monoid encodes information about the arithmetic progressions with step size s contained in a given numerical semigroup S, and has some surprising geometric properties.

Source: *Factorization properties of Leamer monoids* (J. Haarmonn, A. Kalauli, A. Moran, C. O'Neill, R. Pelayo).

Polytopes

(12) *Specializations of Ehrhart reciprocity.* The Ehrhart-Macdonald reciprocity theorem, which relates the Ehrhart function of a polytope to that of its strict interior, can be proven by building up from specialized reciprocity results.

Source: *Computing the continuous discretely* (M. Beck, S. Robins), Chapter 4.

(13) *Inside-out polytopes.* A *graph* is a collection of dots (called *vertices*) and lines between them (called *edges*). A *proper coloring* of a graph is a way to color the vertices so that no two vertices on the same edge are the same color. The number of proper colorings on a graph G using n colors is known as the *chromatic polynomial* of G. As the name suggests, the chromatic polynomial of a graph is a polynomial in n whose degree equals the number of vertices of G.

Chromatic polynomials of graphs can be studied using the Ehrhart functions of what are called *inside-out polytopes*, which are polytopes with the intersection of certain hyperplanes through the middle removed.

Source: *Combinatorial reciprocity theorems* (M. Beck, R. Sanyal), Chapters 1 and 7.

(14) *Flow polytopes.* A *directed graph* is a collection of dots (called *vertices*) and arrow between them (called *edges*). A *flow* on a directed graph is a way to label each edge of G by an element of $\mathbb{Z}$ (or $\mathbb{Z}_n$) so that at each vertex v, the sum of the edges entering v equals the sum of the edges leaving v (i.e. the *in flow* at each vertex equals the *out flow*). The flows on a given directed graph can be counted using a certain family of polytopes, whose integer points each coincide with a flow on G.

Source: *Combinatorial reciprocity theorems* (M. Beck, R. Sanyal), Chapters 1 and 7.

(15) *Order polytopes.* A *partially ordered set* (or *poset* for short) is a set P and a reflexive, antisymmetric, and transitive *partial* ordering $\preceq$ (that is, some elements of P are incomparable under $\preceq$). One of the "standard" examples is a set P of positive integers (say, $\{1, \ldots, 10\}$) under the partial order $a \preceq b$ if $a \mid b$. Indeed, divisibility is reflexive ($a \mid a$), antisymmetric ($a \mid b$ and $b \mid a$ implies $a = b$) and transitive ($a \mid b$ and $b \mid c$ implies $a \mid c$), but some integers are incomparable (e.g. $4 \nmid 6$ and $6 \nmid 4$).

An *order polytope* is a polytope $\mathcal{O}(P)$ constructed from a given finite poset P, and naturally encodes many properties of P. For example, the Ehrhart function of $\mathcal{O}(P)$ counts the number of *order preserving functions* on P.

Source: *Combinatorial reciprocity theorems* (M. Beck, R. Sanyal), Chapters 1, 2, and 6.

(16) *0/1 polytopes.* A 0/1 *polytope* is a polytope in which all vertex coordinates are either 0 or 1. Families of 0/1 polytopes can encode a wide variety of different combinatorics problems, and are often used to demonstrate "extremely bad behavior" that polytopes can exhibit.

Source: *Lectures on* 0/1 *polytopes* (G. Ziegler).

(17) *Counting magic squares.* A *magic square* is an $n \times n$ grid of integers in which all row and column sums coincide. Counting the number of possible magic squares of a given size and row/column sum can be achieved using the Ehrhart functions of certain polytopes, wherein each integer point corresponds to a particular magic square.

Source: *Computing the continuous discretely* (M. Beck, S. Robins), Chapter 6.

(18) *The Hirsch conjecture.* One of the most famous and longstanding open problems in the study of polytopes, the Hirsch conjecture (1957) proposed a bound on the possible distances between vertices in the 1-skeletons of polytopes. Since the first counterexample was discovered in 2012 (with a whopping dimension of 43), several smaller counterexamples have been located, though few (if any) are small enough to write down by hand. Although the conjecture is officially settled, many related questions still remain unanswered.

Source: *Who solved the Hirsch conjecture?* (G. Ziegler).

(19) *The Dehn-Sommerville relations.* The *face numbers* $f_0, f_1, \ldots, f_d$ of a polytope P count the number of faces of P of dimension $0, 1, \ldots, d$, respectively. If P is *simple* (meaning every vertex of P lies in exactly d edges, the smallest number geometrically possible), then the face numbers of P satisfy a collection of equations called the *Dehn-Sommerville relations.* These relations arise frequently in the study of Ehrhart functions of simple polytopes.

Source: *Computing the continuous discretely* (M. Beck, S. Robins), Chapter 5.

(20) *Counting oversemigroups.* A numerical semigroup T is an *oversemigroup* of a numerical semigroup S if $T \supset S$. Every numerical semigroup has only finitely many oversemigroups since there are only finitely many gaps to "fill in". Recently, Ehrhart functions were used in counting the number of oversemigroups by constructing certain polytopes in which each integer point corresponds to an oversemigroup of the given semigroup.

Source: *On the number of numerical semigroups containing two coprime integers* (M. Hellus, R. Waldi).

Generating functions

(21) *Frobenius numbers and partial fractions.* One additional set of "tools" often used to study numerical semigroups comes from complex analysis. For example, the closed formula for the Frobenius number of a 2-generated numerical semigroup can be derived using partial fractions decomposition of rational functions and properties of pole residues.

Source: *Computing the continuous discretely* (M. Beck, S. Robins), Chapter 1.

(22) *Generating functions as a counting tool.* Our use of generating functions in this class (to concisely express certain quasipolynomial functions) only scratches the surface of their utility. Generating functions are used heavily in combinatorics to give concise answers to counting questions when closed forms are difficult or impossible. The techniques for doing so yield an elegant high-level approach to combinatorics problems and some surprisingly slick proofs.

Source: *A walk through combinatorics* (3rd edition) (M. Bóna), Chapter 8.

Advanced algebra topics

(23) *Simplicial homology.* Simplicial complexes (which will appear briefly in the last couple of weeks of the course) are used to construct complicated spaces (such as a sphere or torus) by gluing together basic building blocks called *simplices*. The *homology groups* of a given simplicial complex encode information about the "holes" that appear in the resulting space. For example, a circle has one hole, while a torus (i.e. a hollow donut) has 2 holes: one through the "donut hole" and one on where the "jelly filling" goes.

Source: *Homology theory – a primer* (J. Kun)
`https://jeremykun.com/2013/04/03/homology-theory-a-primer/`

Note: This project requires at least one semester of abstract algebra, but preferably two. At the very least, familiarity with Abelian groups and quotient groups is essential.

(24) *Binomial ideals.* An ideal I in the polynomial ring $\mathbb{Q}[x_1, \ldots, x_d]$ is a *binomial ideal* if it can be generated by differences of monomials. Binomial ideals provide a natural bridge between commutative algebra and semigroups, and are hiding in the background throughout much of the content we will see in this course (in some ways, this is a "4th viewpoint" alongside semigroups, polytopes, and generating functions).

Source: *Combinatorial commutative algebra* (E. Miller, B. Sturmfels), Chapter 7.

Note: This project requires at least one semester of abstract algebra, but preferably two. At the very least, familiarity with ideals, polynomial rings, and quotient rings is essential.

(25) *Hilbert functions.* Hilbert functions, which arise in the study of polynomial ideals, provide a natural bridge between commutative algebra and combinatorics, and are closely related to much of the content we will see in this course. Hilbert's theorem states that certain Hilbert functions are eventually quasipolynomial (this is the connection to quasipolynomial functions in the previous project's "4th viewpoint"). Among other things, Hilbert's theorem can be used to give an algebraic proof of Ehrhart's theorem for integral/rational polytopes.

Source: *Combinatorial commutative algebra* (E. Miller, B. Sturmfels), Chapter 8.

Note: This project requires at least one semester of abstract algebra, but preferably two. At the very least, familiarity with ideals, polynomial rings, and quotient rings is essential.