Spring 2020, Math 621: Week 1 Problem Set
Due: Thursday, February 6th, 2020
Numerical and Affine Semigroups

Discussion problems. The problems below should be worked on in class.

(D1) *Minimal generating sets.* Fix a subsemigroup $S \subset (\mathbb{Z}^d_{\geq 0}, +)$. The goal of this problem is to prove that S has a unique generating set that is minimal with respect to containment.

(a) Determine the unique minimal generating set of each of the following semigroups.

(i) $S = \langle 6, 9, 15, 20, 26, 42, 55 \rangle$

(ii) $S = \langle 11, 15, 19, \ldots, 111 \rangle$

(iii) $S = \langle (0,2), (1,2), (2,0), (2,2), (2,3), (3,2), (3,3) \rangle$

(iv) $S = \{(a,b) \in \mathbb{Z}^2_{\geq 0} : a \leq 3b \text{ and } b \leq 4a\}$

(v) $S = \langle G \rangle$, where $G = \{(a,b) : a \equiv 1 \bmod 3 \text{ and } b \equiv 2 \bmod 3\}$.

(b) Let $S^* = S \setminus \{0\}$. Compare the sets S and $S^* + S^*$ for $S = \langle 6, 9, 20 \rangle$.

(c) Let $\mathcal{A}(S) = S^* \setminus (S^* + S^*)$. Prove that $S = \langle \mathcal{A}(S) \rangle$.

(d) Explain briefly why any generating set for S must contain $\mathcal{A}(S)$.

(e) Prove that if $d = 1$, then the elements of $\mathcal{A}(S)$ must be distinct modulo $m = \min \mathcal{A}(S)$. Conclude that $\mathcal{A}(S)$ must be finite in this case.

(f) Does the previous part necessarily hold if $d = 2$?

(D2) *Geometry of semigroups.* The goal of this problem is to explore several geometric theorems.

(a) It turns out any finitely generated subsemigroup $S = \langle \alpha_1, \ldots, \alpha_k \rangle \subset \mathbb{Z}^r$ is isomorphic to some subsemigroup of $\mathbb{Z}^d_{\geq 0}$, where $d = \dim_{\mathbb{R}} \operatorname{span}_{\mathbb{R}}(S)$ (the *affine dimension* of S).

For each of the following, compute d, and locate an isomorphic subsemigroup of $\mathbb{Z}^d_{\geq 0}$.

(i) $S = \langle (9,6), (15,10), (21,14) \rangle$

(ii) $S = \langle (0,2), (2,1), (-1,2) \rangle$

(iii) $S = \langle (2,0,2), (2,3,5), (4,3,7) \rangle$

(b) *Gordan's Lemma*: given a finite list of linear inequalities with rational coefficients, the subset of $\mathbb{Z}^d$ satisfying them is a finitely generated semigroup.

Determine the (finite) minimal generating set of each of the following.

(i) $S = \{(a,b) \in \mathbb{Z}^2 : 0 \leq 3a \leq 2b\}$

(ii) $S = \{v \in \mathbb{Z}^3 : Av \leq 0\}$, where

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

and each row of $Av \leq 0$ is interpreted as an inequality on v_1, v_2, v_3.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Fix a commutative semigroup $(S, +)$. A relation $\sim$ on S is a *congruence* if (i) $\sim$ is an equivalence relation, and (ii) $\sim$ is closed under *translation* (i.e., $a \sim b$ implies $a + c \sim b + c$ for all $a, b, c \in S$).

(a) Given a (possibly infinite) collection of congruences $\sim_i$, define the *common refinement* $\sim = \bigcap_i \sim_i$ by $a \sim b$ whenever $a \sim_i b$ for all i. Prove that $\sim$ is a congruence.

(b) Prove that if $\sim$ is a congruence on S, then the set $S/\sim$ of equivalence classes of $\sim$ is a semigroup under the operation $[a] + [b] = [a + b]$.

(c) Given a semigroup homomorphism $\varphi : S \to T$, the *kernel* of φ is the relation $\sim = \ker \varphi$ on S setting $a \sim b$ whenever $\varphi(a) = \varphi(b)$. Prove that $\ker \varphi$ is a congruence on S.

(d) State and prove a version of the first isomorphism theorem for semigroups.

(H2) Fix a field $\Bbbk$, and let $R = \Bbbk[x_1, \ldots, x_k]$. In what follows, for $a \in \mathbb{Z}^k_{\ge 0}$, we use the shorthand

$$x^a = x_1^{a_1} \cdots x_k^{a_k}.$$

A *(unital) binomial* is a polynomial of the form $x^a - x^b \in R$ for some $a, b \in \mathbb{Z}^k_{\ge 0}$.

(a) Fix an ideal $I \subset R$. Define a relation $\sim_I$ on $\mathbb{Z}^k_{\ge 0}$ by

$$a \sim_I b \qquad \text{whenever} \qquad x^a - x^b \in I$$

for $a, b \in \mathbb{Z}^k_{\ge 0}$. Prove that $\sim_I$ is a congruence on $\mathbb{Z}^k_{\ge 0}$.

(b) Fix $c_1, \ldots, c_r, d_1, \ldots, d_r \in \mathbb{Z}^k_{\ge 0}$, and let $I = \langle x^{c_1} - x^{d_1}, \ldots, x^{c_r} - x^{d_r} \rangle \subset \Bbbk[x_1, \ldots, x_k]$ (we call I a *binomial ideal*). Prove that the smallest congruence $\sim$ on $\mathbb{Z}^k_{\ge 0}$ satisfying $c_i \sim d_i$ for each i is $\sim = \sim_I$.

Note: "smallest congruence" refers to the congruence with the smallest collection of relations. In particular, the smallest congruence satisfying some property (if it exists) equals the common refinement of all congruences satisfying that property.

(H3) Determine whether each of the following statements is true or false. Prove your assertions.

(a) For any $\alpha_1, \ldots, \alpha_k \in \mathbb{Z}^d_{\ge 0}$ with $d \ge 2$, the set $\langle \alpha_1, \ldots, \alpha_k \rangle \setminus \langle \alpha_1, \ldots, \alpha_{k-1} \rangle$ is infinite.

(b) Given any congruence $\sim$ on $\mathbb{Z}^k_{\ge 0}$, the semigroup $S = \mathbb{Z}^k_{\ge 0}/\sim$ is *cancellative*, that is, $a + c = b + c$ implies $a = b$ for all $a, b, c \in S$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Locate a commutative, cancellative semigroup S such that (i) S is finitely generated, (ii) $0 \in S$ is the only element of S with an inverse, and (iii) S is not isomorphic to any affine semigroup.