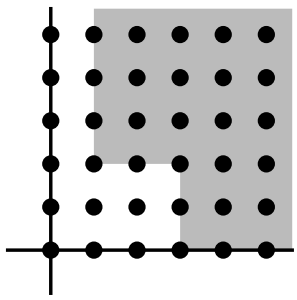
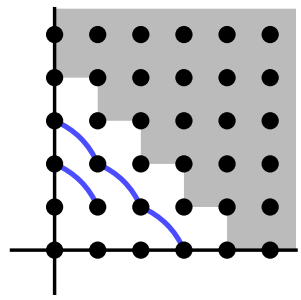
Spring 2020, Math 621: Week 2 Problem Set
Due: Thursday, February 13th, 2020
Graded Rings and Modules

Discussion problems. The problems below should be worked on in class.

(D1) *Combinatorics of monomial and binomial quotients.* The goal of this problem is to make sense of pictures of the following form (called *staircase diagrams*).



$J_1 = \langle x^3, xy^2 \rangle$



$J_2 = \langle x^3 - x^2y, xy - y^2, x^4 \rangle$

(a) Using J_1 above as a guide, draw the staircase diagram for $I = \langle x^3y, x^3y^3, xy^3 \rangle$.

(b) For an ideal I with monomial and binomial generators, we add *edges* to the staircase diagram so each connected component corresponds to an equivalence class modulo $\sim_I$. Locate the largest monomial ideal contained in J_2.

(c) Using J_2 above as a guide, draw the staircase diagram of each of the following ideals I, and find $\dim_{\Bbbk} R/I$.

(i) $I = \langle x - y, x^3 \rangle$
(ii) $I = \langle y(x^2 - 1), xy^2 - y^2, y^3 \rangle$
(iii) $I = \langle x^3 - 1, y(x^2 - 1), y^2(x - 1), y^3 \rangle$
(iv) $I = \langle x^2 - y^2, x^4 - y^3 \rangle$

(d) Draw the staircase diagram for $I = \langle x^3, y^2, z^2, xyz \rangle \subset \Bbbk[x, y, z]$ (yes, a 3D picture).

(e) Does there exist an ideal $J \supsetneq I$, generated by monomials and binomials, for which I is the largest monomial ideal contained in J?

(D2) *Computing Hilbert functions.* Let $R = \Bbbk[x, y]$, graded by total degree. Determine the Hilbert function of each of the following graded modules M. Your answer to each should be a (possibly piecewise) formula for $\mathcal{H}(M; t)$ in terms of t.

(a) $M = I$, where $I = \langle x^2y, xy^2 \rangle \subset R$

(b) $M = R/I$, where $I = \langle x^2y, xy^2 \rangle \subset R$

(c) $M = R/I$, where $I = \langle x^3, y^3 \rangle \subset R$

(d) $M = R/I$, where $I = \langle x^2 - y^2 \rangle \subset R$

(e) $M = R \oplus (R/I)$, where $I = \langle x^2 + xy + y^2 \rangle \subset R$

(f) $M = (R \oplus R)/N$, where $N = \langle x^2e_1 - xye_1, xye_2 - y^2e_2 \rangle \subset R \oplus R$
(here, $e_1 = (1, 0)$ and $e_2 = (0, 1)$ are the standard basis of $R \oplus R$)

(g) $M = (R \oplus R)/N$, where $N = \langle x^2e_1 - xye_2, xye_1 - y^2e_2 \rangle \subset R \oplus R$

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) For each function h below, find a graded ring R and a finitely generated graded module M over R with $\mathcal{H}(M;t) = h(t)$ for all t in the domain of h.

(a) $h : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ given by $h(t) = 5t + 7$

(b) $h : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ given by

$$h(t) = \begin{cases} t - 7 & \text{if } t \geq 10; \\ 3 & \text{if } t = 6; \\ 0 & \text{otherwise.} \end{cases}$$

(c) $h : \mathbb{Z}_{\geq 0}^2 \to \mathbb{Z}_{\geq 0}$ given by

$$h(t_1, t_2) = \begin{cases} 2 & \text{if } t_1 = 0 \text{ or } t_2 = 0; \\ t_1 + t_2 & \text{otherwise.} \end{cases}$$

(H2) Fix an ideal $I \subset R = \Bbbk[x_1, \ldots x_k]$. Develop a criterion for when I is homogeneous under **all** $\mathbb{Z}$-gradings of R (that is to say, for any choices of $\deg(x_1), \ldots, \deg(x_k) \in \mathbb{Z}_{\geq 0}$, the ideal I is homogeneous).

(H3) Determine whether each of the following statements is true or false. Prove your assertions.

(a) Any function $h : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ occurs as the Hilbert function of some graded module over a polynomial ring.

(b) If $I \subset J$ are homogeneous ideals in $R = \Bbbk[x_1, \ldots, x_k]$ (under the standard grading) and $\mathcal{H}(R/I;t) = \mathcal{H}(R/J;t)$, then $I = J$.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Prove or disprove: if $I \subset J$ are homogeneous ideals in $R = \Bbbk[x_1, \ldots, x_k]$ (under the standard grading) and $\sim_I = \sim_J$, then $I = J$.