Spring 2020, Math 621: Week 3 Problem Set
Due: Thursday, February 20th, 2020
Rational Generating Functions

Discussion problems. The problems below should be worked on in class.

(D1) *Power series of quasipolynomial functions.* Recall that in lecture, we saw

$$1 + 2z + 3z^2 + \cdots = \sum_{n=0}^{\infty}(n+1)z^n = \frac{1}{(1-z)^2},$$

and that the "formal derivative" of $A(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ is

$$A'(z) = \tfrac{d}{dz}A(z) = a_1 + 2a_2 z + 3a_3 z^2 + \cdots = \sum_{n=0}^{\infty}(n+1)a_{n+1}z^n.$$

(a) Manipulate the first expression to write $\sum_{n=0}^{\infty} nz^n$ as a rational expression in z.
(b) Use formal differentiation to write $\sum_{n=0}^{\infty} n^2 z^n$ as a rational expression in z.
(c) Use formal differentiation to write $\sum_{n=0}^{\infty} n^3 z^n$ as a rational expression in z.

(D2) *Multivariate power series.* In this problem, we will explore a geometric interpretation of rational power series in the ring $\mathbb{Q}[\![z_1, z_2]\!]$.

(a) Using power series multiplication, find all nonzero terms in

$$A(z) = \frac{1}{(1-z_1^3 z_2)(1-z_2^2)}$$

with total degree at most 10. Plot their exponents as points in $\mathbb{R}^2$.

(b) Do the same for the power series

$$B(z) = \frac{1}{(1-z_1^2)(1-z_1 z_2)(1-z_2^2)}.$$

Label each point with its coefficient in $A(z)$. What does this appear to coincide with?

(c) Find a rational expression for the formal power series

$$C(z) = \sum_{(a,b)\in S} z_1^a z_2^b$$

for each of the following sets $S \subset \mathbb{Z}^2_{\geq 0}$.

(i) $S = \langle (0,2), (1,1), (0,2) \rangle$
(ii) $S = \{(a,b) \in \mathbb{Z}^2_{\geq 0} : 2a \geq b\}$
(iii) $S = \{(a,b) \in \mathbb{Z}^2_{\geq 0} : 2a \geq b \text{ and } a \geq 2\}$
(iv) $S = \{(a,b) \in \mathbb{Z}^2_{\geq 0} : x^a y^b \in I\}$, where $I = \langle x^3, x^2 y, y^2 \rangle \subset \Bbbk[x,y]$
(v) $S = \{(a,b) \in \mathbb{Z}^2_{\geq 0} : x^a y^b \notin I\}$, where $I = \langle x^3, x^2 y, y^2 \rangle \subset \Bbbk[x,y]$
(vi) $S = \{(a,b) \in \mathbb{Z}^2_{\geq 0} : \mathcal{H}(R/I; a, b) \neq 0\}$, where $I = \langle x_1^2 - x_2^2, x_3^3 \rangle \subset R = \Bbbk[x_1, x_2, x_3]$ with $\deg(x_1) = \deg(x_2) = (1,0)$ and $\deg(x_3) = (0,1)$

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Find a rational expression for the formal power series

$$C(z) = \sum_{(a,b)\in S} z_1^a z_2^b,$$

where $S = \{(a,b) \in \mathbb{Z}^2_{\geq 0} : a \leq 2b,\ b \leq 3a+1, \text{ and } a+b \geq 3\} \subset \mathbb{Z}^2_{\geq 0}$.

(H2) Fix power series $A(z) = \sum_{n=0}^{\infty} a_n z^n$ and $B(z) = \sum_{n=0}^{\infty} b_n z^n$.

(a) Prove that

$$\frac{d}{dz}\big(A(z)B(z)\big) = A'(z)B(z) + A(z)B'(z).$$

(b) Prove that if $b_0 \neq 0$, then

$$\frac{d}{dz}\left(\frac{1}{B(z)}\right) = -\frac{B'(z)}{B(z)^2}.$$

(c) Conclude that if $b_0 \neq 0$, then

$$\frac{d}{dz}\left(\frac{A(z)}{B(z)}\right) = \frac{A'(z)B(z) - A(z)B'(z))}{B(z)^2}.$$

Hint: parts (b) and (c) can be done without writing any sigma sums!

(H3) Suppose $f : \mathbb{Z}_{\geq 0} \to \mathbb{Q}$ is a function, $h(z)$ is a power series, $d \geq 1$ and $k \geq 0$ are integers, and

$$\sum_{n=0}^{\infty} f(n) z^n = \frac{h(z)}{(1-z^d)^{k+1}}.$$

(a) Prove that for any $k \geq 0$,

$$\sum_{n=0}^{\infty} n^k z^n = \frac{h_k(z)}{(1-z)^{k+1}}$$

for some polynomial $h_k(z)$ of degree k.

(b) Prove that if f is eventually quasipoynomial of degree at most k and period d, then h is a polynomial.

(c) Prove that if h is a polynomial, then f is eventually quasipolynomial of degree at most k and period dividing d.

(d) Suppose f is eventually quasipoynomial of degree k and period d, and h is a polynomial. The *dissonance point* of f is the smallest integer D such that $f\mid_{\geq D}$ is quasipolynomial. Find a relationship between D the degree of h.

Challenge problems. Challenge problems are not required for submission, but bonus points will be awarded for submitting a partial attempt or a complete solution.

(C1) Characterize which functions $f : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ satisfy

$$\sum_{n\geq 0} f(n) z^n = \frac{h(z)}{g(z)}$$

for some polynomials $h(z)$ and $g(z)$ with coefficients in $\mathbb{C}$.