Spring 2020, Math 621: Week 4 Problem Set
Due: Thursday, February 27th, 2020
Hilbert Series and Hilbert's Theorem

Discussion problems. The problems below should be worked on in class.

(D1) *Hilbert series of monomial ideals.*

(a) Find $\operatorname{Hilb}(M; z_1, z_2)$ for each of the finely-graded modules M over $R = \Bbbk[x, y]$ below.

(i) $M = I$, where $I = \langle xy^2 \rangle$
(ii) $M = I$, where $I = \langle x^3, xy^2, y^4 \rangle$
(iii) $M = R/I$, where $I = \langle x^3, xy^2, y^4 \rangle$
(iv) $M = R/I$, where $I = \langle x^4, x^3y, x^2y^2, y^4 \rangle$

(b) Find $\operatorname{Hilb}(M; z)$ for each of the above modules M (under the standard grading).

(c) Conjecture a closed form for $\operatorname{Hilb}(R/I; z_1, z_2)$, where

$$I = \langle x^{a_1}y^{b_1}, \ldots, x^{a_k}y^{b_k} \rangle \subset R = \Bbbk[x, y]$$

is a monomial ideal with $a_1 > \cdots > a_k$ and $b_1 < \cdots < b_k$.

(d) Locate a monomial ideal $I \subset \Bbbk[x, y]$ for which, under the standard grading,

$$\operatorname{Hilb}(I; z) = \frac{3z^5 - z^6 + z^7 - 2z^8}{(1 - z)^2}.$$

(e) Find $\operatorname{Hilb}(R/I; z_1, z_2, z_3)$, where $I = \langle x^2, y^2, z^2, xyz \rangle \subset R = \Bbbk[x, y, z]$.
Hint: draw the staircase diagram (yes, in 3D).

(f) Locate a monomial ideal $I \subset \Bbbk[x_1, x_2, x_3]$ for which, under the standard grading,

$$\operatorname{Hilb}(I; z) = \frac{6z^2 - 8z^3 + 3z^4}{(1 - z)^3}.$$

(D2) *Computing Hilbert series.* Find a rational form for the Hilbert series $\operatorname{Hilb}(M; z)$ of each of the following (standard graded) modules M over $R = \Bbbk[x, y]$.

(a) $M = R \oplus R$

(b) $M = R/I$, where $I = \langle x^2 - y^2 \rangle \subset R$

(c) $M = R \oplus (R/I)$, where $I = \langle x^2 + xy + y^2 \rangle \subset R$

(d) $M = R/I \oplus R/J$, where $I = \langle x^3 - x^2y, xy - y^2 \rangle$ and $J = I + \langle x^4 \rangle$

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Locate a **single** graded ring R and R-modules M_1, M_2, and M_3 with the Hilbert series specified below.

(a) $\mathrm{Hilb}(M_1; z) = \dfrac{1}{(1 - z^3)(1 - z^5)}$

(b) $\mathrm{Hilb}(M_2; z) = \dfrac{z^5 + 3z^7}{(1 - z^2)(1 - z^3)^2}$

(c) $\mathrm{Hilb}(M_3; z) = \dfrac{2z^5 - z^7}{(1 - z^2)(1 - z^3)}$

(H2) The *Hilbert series* of a numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subset \mathbb{Z}_{\geq 0}$ is

$$\mathrm{Hilb}(S; z) = \sum_{a \in S} z^a.$$

(a) Prove **using Hilbert's Theorem** that

$$\mathrm{Hilb}(S; z) = \frac{g(z)}{(1 - z^{n_1}) \cdots (1 - z^{n_k})}$$

for some polynomial $g(z)$.

(b) Fix $m \in S$. Prove that

$$\mathrm{Hilb}(S; z) = \frac{a(z)}{1 - z^m}$$

for some polynomial $a(z)$ with positive integer coefficients.

(H3) Determine whether each of the following statements is true or false. Prove your assertions.

(a) If $I \subset R = \Bbbk[x_1, \ldots, x_k]$ is a monomial ideal, and we write

$$\mathrm{Hilb}(R/I; z_1, \ldots, z_k) = \frac{h(z_1, \ldots, z_k)}{(1 - z_1) \cdots (1 - z_k)} \qquad \text{and} \qquad \mathrm{Hilb}(R/I; z) = \frac{g(z)}{(1 - z)^k}$$

with respect to the fine grading and the standard grading, respectively, then h and g have the same number of nonzero terms.

(b) There exists a monomial ideal $I \subset \Bbbk[x_1, x_2, x_3]$ such that, under the standard grading,

$$\mathrm{Hilb}(I; z) = \frac{3z^6 - 4z^9 + z^{11}}{(1 - z)^3}.$$