Spring 2020, Math 621: Week 6 Problem Set
Due: Thursday, March 12th, 2020
Polytopes and Polyhedra

Discussion problems. The problems below should be worked on in class.

(D1) *Warmup.* Consider the following polytope.

$$P = \text{conv}\{(1,0,0),\ (0,1,0),\ (-1,-1,0),\ (0,0,1),\ (0,0,-1)\}$$

(a) Draw P (as best you can).

(b) Find a matrix A so that the system of inequalities $Ax \le \vec{1}$ completely describe $x \in P$. Is your matrix A unique?

(c) Draw the face lattice of P.

(D2) *Proving things about polytopes.* The goal of this problem is to get a feeling for how to write rigorous proofs involving polytopes.

(a) Draw the cubes $C_2 \subset \mathbb{R}^2$ and $C_3 \subset \mathbb{R}^3$. Label the vertices in each drawing.

(b) Formulate a conjecture on when two vertices v_1 and v_2 of the d-dimensional cube C_d are connected by an edge.

The goal of the remainder of this problem is to prove your conjecture. Be **thorough** and **rigorous** in your arguments. Use C_2 and C_3 as a starting place.

(c) For each edge e connecting vertices v_1 and v_2 of C_d, find an equation of a hyperplane H (which should have the form $a_1x_1 + \cdots + a_dx_d = b$ for some $a_1, \ldots, a_d, b \in \mathbb{Q}$) so that (i) the only vertices H contains are v_1 and v_2, and (ii) the remaining vertices of C_d lie on the same side of H. This ensures H is "just touching the polytope" at e.

Hint: be systematic, and use symmetry to your advantage!

(d) For two points $x, y \in \mathbb{R}^d$, let

$$\overline{xy} = \{tx + (1-t)y : 0 \le t \le 1\}$$

denote the line segment connecting x and y. For each pair of vertices v_1 and v_2 **not** connected by an edge, locate two points w_1 and w_2 in C_d **not** in $\overline{v_1v_2}$ for which $\overline{w_1w_2} \cap \overline{v_1v_2}$ is nonempty. Briefly explain why this proves $\overline{v_1v_2}$ is not an edge.

Hint: it is possible to choose each w_1 and w_2 to also be vertices of C_d (this is special to the cube and is **not** true in general).

(e) Locate an example illustrating the caveat of the hint in the previous part.

(f) Locate a hyperplane demonstrating each vertex of C_d is indeed a vertex.

(g) How might you prove that there are no other vertices?

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Consider the polytope

$$P = \text{conv}\{(0,0,0),\ (0,0,1),\ (0,1,0),\ (0,1,1),\ (1,0,0),\ (1,1,0)\},$$

which is a cube with 2 adjacent vertices removed.

(a) Draw P (as best you can).

(b) Find the H-description of P.

(c) Draw the face lattice of P.

(H2) Prove, from first principles, that $\text{conv}\{0,1\}^d = [0,1]^d$.

(H3) Prove that any two vertices of the d-simplex $S_d = \text{conv}\{0, e_1, \ldots, e_d\}$ share an edge.

(H4) The *permutohedron* is the polytope P_n whose vertices consist of all possible orderings of the coordinates of $(1, 2, \ldots, n)$. For example,

$$P_3 = \{(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)\}.$$

Prove that $\dim P_n = n - 1$.

Hint: to prove $\dim P_n \leq n - 1$, find an affine hyperplane that contains P_n. To prove $\dim P_n \geq n - 1$, locate $n - 1$ linearly independent vectors of the form $v - w$ for $v, w \in P_n$.

(H5) Determine whether each of the following statements is true or false. Prove your assertions.

(a) Among polytopes with f-vector $(1, V, E, F, 1)$, the value $F - V$ can be arbitrarily large.

(b) The subsets $P_1 = [0,1]$ and $P_2 = (0,1)$ of $\mathbb{R}$ are both 1-dimensional polytopes.

Announcement. For those who are want to avoid drawing polytopes by hand, there is a free (web-based) program you can use, developed by Nils Olsson (an SDSU student from Math 596).

`https://nilsso.github.io/pages/math/semi-comb/polytope.html`